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A Course in Differential Geometry

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ABSTRACT. This book provides an introduction to differential geometry, with principal emphasis on Riemannian geometry. It covers the essentials, concluding with a chapter on the Yamabe problem, which shows what research in the field looks like. It is a textbook, at a level which is accessible to graduate students. Its aim is to facilitate the study and the teaching of differential geometry. It is teachable on a chapter-by-chapter basis. Many problems and a number of solutions are included; most of them extend the course itself, which is confined to the main topics, such as: differential manifolds, submanifolds, differential mappings, tangent vectors, differential forms, orientation, manifolds with boundary, Lie derivative, integration of p-direction field, connection, torsion, curvature, geodesics, covariant derivative, Riemannian manifolds, exponential mapping, and spectrum.
A mon Professeur
André Lichnerowicz
(in memoriam)
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Preface

This book provides an introduction to differential geometry, with principal emphasis on Riemannian geometry. It can be used as a course for second-year graduate students. The main theorems are presented in complete detail, but the student is expected to provide the details of certain arguments. We assume that the reader has a good working knowledge of multidimensional calculus and point-set topology.

Many readers have been exposed to the elementary theory of curves and surfaces in three-space, including tangent lines and tangent planes. But these techniques are not necessary prerequisites for this book.

In this book we work abstractly, so that the notion of tangent space does not necessarily have a concrete realization. Nevertheless we will eventually prove Whitney's theorem asserting that any abstract n-dimensional manifold may be imbedded in the Euclidean space $\mathbb{R}^p$ if $p$ is sufficiently large.

In order to develop the abstract theory, one must work hard at the beginning, to develop the notion of local charts, change of charts, and atlases. Once these notions are understood, the subsequent proofs are much easier, allowing one to obtain great generality with maximum efficiency. For example, the proof of Stokes' theorem—which is difficult in a concrete context—becomes transparent in the abstract context, reducing to the computation of the integral of a derivative of a function on a closed interval of the real line.

In Chapter I we find the first definitions and two important theorems, those of Whitney and Sard.

Chapter II deals with vector fields and differential forms.
Chapter III concerns integration of vector fields, then extends to $p$-plane fields. We cite in particular the interesting proof of the Frobenius theorem, which proceeds by mathematical induction on the dimension.

Chapter IV deals with connections, the most difficult notion in differential geometry. In Euclidean space the notion of parallel transport is intuitive, but on a manifold it needs to be developed, since tangent vectors at distinct points are not obviously related. Loosely speaking, a connection defines an infinitesimal direction of motion in the tangent bundle, or, equivalently, a connection defines a sort of directional derivative of a vector field with respect to another vector. This concrete notion of connection is rarely taught in books on connections. In our work we devote ten pages to developing these ideas, together with the related notions of torsion, curvature and a working knowledge of the covariant derivative. All of these notions are essential to the study of real or complex manifolds.

In Chapter V we specialize to Riemannian manifolds. The viewpoint here is to deduce global properties of the manifold from local properties of curvature, the final goal being to determine the manifold completely.

In Chapter VI we explore some problems in partial differential equations which are suggested by the geometry of manifolds.

The last three chapters are devoted to global notation, specifically to using the covariant derivative instead of computing in local coordinates with partial derivatives. In some cases we are able to reduce a page of computation in local coordinates to just a few lines of global computation. We hope to further encourage the use of global notation among differential geometers.

The aim of this book is to facilitate the teaching of differential geometry. This material is useful in other fields of mathematics, such as partial differential equations, to name one. We feel that workers in PDE would be more comfortable with the covariant derivative if they had studied it in a course such as the present one. Given that this material is rarely taught, one may ask why? We feel that it requires a substantial amount of effort, and there is a shortage of good references. Of course there are reference books such as Kobayashi and Nomizu [5], which can be consulted for specific information, but that book is not written as a text for students of the subject.

The present book is made to be teachable on a chapter-by-chapter basis, including the solution of the exercises. The exercises are of varying difficulty, some being straightforward or solved in existing literature; others are more challenging and more directly related to our approach.

This book is an outgrowth of a course which I presented at the Université Paris VI. I have included many problems and a number of solutions. Some of these originated from examinations in the course. I am very grateful to my friend Mark Pinsky, who agreed to read the manuscript from beginning
to end. His comments allowed me to make many improvements, especially in the English. I would like to thank also one of my students, Sophie Bismuth, who helped me to prepare the final draft of this book.
Background Material

In this chapter we recall some fundamental knowledge which will be used in the book: topology, algebra, integration, and differential calculus.

Topology

0.1. Definition. A topology on a set $E$ is defined by a family $\mathcal{O}$ of subsets of $E$, called open sets, such that

a) The set $E$ and the null set $\emptyset$ are open sets.
b) Any union of open sets is an open set.
c) Any finite intersection of open sets is an open set.

$(E, \mathcal{O})$ is a topological space.

0.2. Examples. If $\mathcal{O} = \{E, \emptyset\}$, the corresponding topology is called trivial. If $\mathcal{O}$ consists of all subsets of $E$, the topology is called the discrete topology.

On $\mathbb{R}^n$ the usual topology may be defined as follows: Let $x$ be a point of $\mathbb{R}^n$ and $\rho > 0$ a real number. We consider the open ball of center $x$ and radius $\rho$, $B_x(\rho) = \{y \in \mathbb{R}^n \mid \|x - y\| < \rho\}$. An open set in $\mathbb{R}^n$ will be a union of open balls or the empty set $\emptyset$.

0.3. Induced topology. Let $F$ be a subset of $E$ endowed with a topology $\mathcal{O}$. The induced topology on $F$ is defined by the following set $\tilde{\mathcal{O}}$ of subsets of $F$: $\tilde{A} \in \tilde{\mathcal{O}}$ if and only if $\tilde{A} = A \cap F$ with $A \in \mathcal{O}$.

0.4. Example. Let $F$ be a finite set of points in $\mathbb{R}^n$. The topology on $F$ induced by the usual topology on $\mathbb{R}^n$ is the discrete topology. We will find other examples in 1.16.
0.5. **Definitions.** A *neighbourhood* of a point \( x \) in a topological space \( E \) is a subset of \( E \) containing an open set which contains the point \( x \).

We can verify that a subset \( A \subset E \) is open if and only if it is a neighbourhood of each of its points.

\( B \subset E \) is *closed* if \( A = E \setminus B \) is open.

A topological space \( E \) is said to be *connected* if the only subsets which are both open and closed are the empty set \( \emptyset \) and the space \( E \) itself.

The *closure* \( \overline{B} \) of a subset \( B \subset E \) is the smallest closed set containing \( B \). The closure \( \overline{B} \) always exists—indeed, the intersection of all closed sets which contain \( B \) (\( E \) is one of them) is a closed set according to b) in 0.1.

0.6. **Proposition.** Any neighbourhood \( A \subset B \) has a nonempty intersection with \( B \).

**Proof.** Let \( \Omega \subset A \) be an open neighbourhood of \( x \). If \( \Omega \cap B = \emptyset \), then \( E \setminus \Omega \) is a closed set containing \( B \); hence \( B \subset E \setminus \Omega \) and \( x \not\in B \), a contradiction.

0.7. **Definitions.** The *interior* \( \overset{\circ}{B} \) of \( B \subset E \) is the largest open set contained in \( B \) (\( \overset{\circ}{B} \) is the union of all open sets included in \( B \).)

A topological space is *separable* if it has a countable basis of open sets \( \{A_i\}_{i \in \mathbb{N}} \). That means any neighbourhood of \( x \) contains at least one \( A_i \) with \( x \in A_i \).

A topological space is *Hausdorff* if any two distinct points have disjoint neighbourhoods.

A family \( \{\Omega_i\}_{i \in I} \) of subsets of \( E \) is a *covering* of \( B \subset E \) if \( B \subset \bigcup_{i \in I} \Omega_i \).

A *subcovering* of this covering is a subset of the family, \( \{\Omega_i\}_{i \in J} \) (with \( J \subset I \)), which itself is a covering. If \( J \) is finite the subcovering is said to be finite.

0.8. **Definition.** A subset \( A \subset E \) is a *compact set* if it is Hausdorff and if any covering of \( A \) by open sets has a finite subcovering.

This definition implies the following necessary and sufficient condition: \( A \subset E \), a Hausdorff topological space, is compact if and only if any family of closed sets whose intersection is empty has a finite subfamily of empty intersection.

0.9. **Theorem.** Let \( E \) be a Hausdorff topological space. If \( K \subset E \) is a compact set, \( K \) is closed. This condition is sufficient when \( E \) is compact.

**Proof.** We argue by contradiction. If \( K \) is not closed (\( K \neq \overline{K} \)), there exists \( x \in \overline{K} \) such that \( x \not\in K \). Now in a Hausdorff topological space, the intersection of the closed neighbourhoods of a point \( x \) is just the subset \( \{x\} \), which is closed. Indeed, for any \( y \in E \) there exist disjoint open sets \( \Theta \) and
\( \Omega \), neighbourhoods respectively of \( x \) and \( y \). \( E \setminus \Omega \) is a closed set, which is a neighbourhood of \( x \) since \( \Theta \subset E \setminus \Omega \), and \( y \notin E \setminus \Omega \). Thus the traces on \( K \) of the closed neighbourhoods \( \{V_i\}_{i \in I} \) of \( x \) would have an empty intersection. So \( \{E \setminus V_i\}_{i \in I} \) would be a covering by open sets of \( K \).

Since \( K \) is compact, there would exist a finite set \( J \subset I \) such that \( \{E \setminus V_i\}_{i \in J} \) is a covering of \( K \). Thus \( \bigcap_{i \in J} V_i = V \) would be a neighbourhood of \( x \) and \( V \cap K \neq \emptyset \). Since \( x \in K \), Proposition 0.6 gives a contradiction.

Suppose \( E \) compact and \( A \subset E \) closed. Then the closed sets for \( A \) are closed sets for \( E \), and compactness for \( A \) follows from the necessary and sufficient condition for \( A \) to be compact (see above).

0.10. Definition. Let \( E \) and \( F \) be two topological spaces. A map \( f \) of \( E \) into \( F \) is continuous if the preimage \( f^{-1}(\Omega) \) of any open set \( \Omega \subset F \) is an open subset of \( E \).

0.11. Theorem. The image by a continuous map of a compact set is compact.

Proof. Let \( K \subset E \) be a compact set. Consider any covering of \( f(K) \) by open sets \( \Omega_i \) (\( i \in I \)). \( \{f^{-1}(\Omega_i)\}_{i \in I} \) is a covering of \( K \) by open sets; thus there exists a finite set \( J \subset I \) such that \( K \subset \bigcup_{i \in J} f^{-1}(\Omega_i) \). So \( \{\Omega_i\}_{i \in J} \) is a covering of \( f(K) \).

0.12. Definitions. A continuous map is said to be proper if the preimage of every compact set is a compact set.

If \( E \) and \( F \) are Hausdorff topological spaces, a continuous map \( f \) of \( E \) into \( F \) is proper if \( E \) is compact. Indeed, let \( K \subset F \) be a compact set. Since \( K \) is a closed set, \( f^{-1}(K) \) is a closed set. So \( f^{-1}(K) \) is compact, since a closed set in a compact set is a compact set (Theorem 0.9).

Let \( E \) and \( F \) be two topological spaces. A map \( f \) of \( E \) onto \( F \) is a homeomorphism if it is one to one and if \( f \) and \( f^{-1} \) are continuous maps.

Tensors

0.13. Definition. Let \( E \) and \( F \) be two vector spaces of dimension respectively \( n \) and \( p \). The tensor product of \( E \) and \( F \) is a vector space of dimension \( np \), and is denoted by \( E \otimes F \). A vector of \( E \otimes F \) is called a tensor. To \( x \in E \) and \( y \in F \) we associate \( x \otimes y \in E \otimes F \). This product has the following properties:

a) \( (x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y \) and \( x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2 \), where \( x, x_1, x_2 \) belong to \( E \) and \( y, y_1, y_2 \) belong to \( F \).

b) If \( \alpha \in \mathbb{R} \) (in this course the vector spaces are on \( \mathbb{R} \)), then

\[ (\alpha x) \otimes y = x \otimes (\alpha y) = \alpha (x \otimes y). \]
c) If \( \{e_i\}_{1 \leq i \leq n} \) is a basis of \( E \) and \( \{f_j\}_{1 \leq j \leq p} \) a basis of \( F \), then \( e_i \otimes f_j \) is a basis of \( E \otimes F \).

d) If \( G \) is a third vector space, with \( E \otimes (F \otimes G) = (E \otimes F) \otimes G \), then 
\[
(x \otimes y) \otimes z = x \otimes (y \otimes z),
\]
for \( z \) belonging to \( G \).

The tensor product is associative; it is not commutative. Let \( x \in E \) and \( y \in F \), \( x = x^i e_i \) and \( y = y^j f_j \). Here we use

**The Einstein Convention:** When the same index (such as \( i \) or \( j \)) is above and below, summation over this index is assumed (here from 1 to \( n \) for \( i \), and from 1 to \( p \) for \( j \)). \( i \) (or \( j \)) is called a dummy index.

According to a), \( x \otimes y = x^i y^j e_i \otimes f_j \), \( t^{ij} = x^i y^j \) are the components of \( x \otimes y \) in the basis \( \{e_i \otimes f_j\} \) of \( E \otimes F \).

**What happens when we change basis?**

Let \( \{\tilde{e}_\alpha\}_{1 \leq \alpha \leq n} \) and \( \{\tilde{f}_\beta\}_{1 \leq \beta \leq p} \) be other bases respectively of \( E \) and \( F \).
Then \( \tilde{e}_\alpha = a^\alpha_i e_i \) and \( \tilde{f}_\beta = c^\beta_j f_j \). Or, if we want, \( e_i = b^i_\alpha \tilde{e}_\alpha \) and \( f_j = d^j_\beta \tilde{f}_\beta \); here \((b^\alpha_\beta)\) is the inverse matrix of \((a^\alpha_\beta)\), and \((d^\beta_\alpha)\) is the inverse matrix of \((c^\beta_\alpha)\).

A tensor \( T = t^{\alpha \beta} \tilde{e}_\alpha \otimes \tilde{f}_\beta = t^{ij} e_i \otimes f_j \). Thus
\[
t^{ij} = t^{\alpha \beta} a^\beta_i d^j_\beta \quad \text{and} \quad t^{\alpha \beta} = t^{ij} b^\alpha_i d^j_\beta.
\]

### 0.14. Definition

A \((p, q)\)-tensor, attached to a vector space \( E \) of dimension \( n \), is a tensor of \( E_1 \otimes E_2 \otimes \cdots \otimes E_{p+q} \), where \( E_i = E \) for \( q \) values of \( i \) and \( E_j = E^* \), the dual space of \( E \), for \( p \) values of \( j \). We say that the tensor is \( p \) times **covariant** and \( q \) times **contravar cant**. We denote by \( \otimes E^* \otimes E \) the set of the \((p, q)\)-tensors.

Let \( \{\theta^j\}_{1 \leq j \leq n} \) be the dual basis of \( \{e_i\}_{1 \leq i \leq n} \). A \((p, q)\)-tensor is a linear combination of \( f_1 \otimes f_2 \otimes \cdots \otimes f_{p+q} \), where \( \tilde{f}_j = \theta^j \) for \( p \) values of \( j \), and \( f_i = e_i \) for \( q \) values of \( i \). If we consider another basis of \( E \), say \( \{\tilde{e}_\alpha\}_{1 \leq \alpha \leq n} \), \( \tilde{e}_\alpha = a^\alpha_i e_i \), the new dual basis \( \{\tilde{\theta}^\beta\}_{1 \leq \beta \leq p} \) is given by \( \tilde{\theta}^\beta = b^\beta_j \theta^j \), where \((b^\beta_\gamma)\) is the inverse matrix of \((a^\alpha_\beta)\): \( b^\beta_\gamma a^\alpha_\beta = \delta^\beta_\gamma \), the Kronecker tensor \((\delta^\beta_\gamma = 0 \text{ if } i \neq j \text{ and } \delta^\beta_j = 1 \text{ if } i = j \)).

### 0.15. Examples

A \((1, 0)\)-tensor \( \omega \) is a 1-form: \( \omega = \omega_j \theta^j = \tilde{\omega} \tilde{\theta}^\beta \); thus \( \omega_j = b^\beta_\beta \tilde{\omega} \). The components of \( \omega \) are \( \omega_j \) in the basis \( \{\theta^j\} \) and \( \tilde{\omega} \) in the basis \( \{\tilde{\theta}^\beta\} \). The indices \( j \) and \( \beta \) are called **covariant indices**.

A \((0, 1)\)-tensor \( X \) is a vector: \( X = X^i e_i = \tilde{X}^\alpha \tilde{e}_\alpha \). Thus \( X^i = a^\alpha_i \tilde{X}^\alpha \). The components of \( X \) are \( X^i \) in the basis \( \{e_i\} \) and \( \tilde{X}^\alpha \) in the basis \( \{\tilde{e}_\alpha\} \). The indices \( i \) and \( \alpha \) are called **contravariant indices**.
The contravariant indices are up, and the covariant indices are down.

When we change the basis of $E$, $\{e_i\} \rightarrow \{\tilde{e}_\alpha\}$, we express the components of a $(p, q)$-tensor $T$ (in the basis $\{e_i\}$) in terms of its new components (in the basis $\{\tilde{e}_\alpha\}$) by means of the matrix $((b^j_i))$ for the covariant indices and the matrix $((a^i_\alpha))$ for the contravariant indices.

**0.16. Examples.** In this book we will consider a Riemannian metric $g$. At a point, $g$ is a $(2,0)$-tensor on the tangent space which is isomorphic to $\mathbb{R}^n$. $g_{ij}$ are the components of $g$ in the basis $\{e_i\}$. If $g_{\alpha\beta}$ are the components in a new basis $\{\tilde{e}_\alpha\}$, we get $g_{ij} = b^\alpha_i b^\beta_j g_{\alpha\beta}$, since $g = g^{ij} \partial_i \otimes \partial^j = g_{\alpha\beta} \tilde{\partial}_\alpha \otimes \tilde{\partial}_\beta$.

We will consider also the curvature tensor $R$, which is a $(3,1)$-tensor. We get

$$R_{ij}^k = b^\alpha_i a^j_\beta b^\gamma_k b^\beta_i \tilde{R}_{\alpha\gamma\delta},$$

$R_{ij}^k$ being the components of $R$ in the first basis and $\tilde{R}_{\alpha\beta\gamma\delta}$ the components in the new basis.

**0.17. Theorem.** A system of $n^{p+q}$ real numbers $t_{i_1i_2\ldots i_p}{}^{i_{p+1}\ldots i_{p+q}}$ attached to a basis $\theta^{i_1} \otimes \theta^{i_2} \otimes \ldots \otimes \theta^{i_p} \otimes e_{i_{p+1}} \otimes \ldots \otimes e_{i_{p+q}}$ are the components of a $(p, q)$-tensor on $E$ if and only if in a new basis $\tilde{\theta}^{a_1} \otimes \ldots \otimes \tilde{\theta}^{a_p} \otimes \tilde{e}_{a_{p+1}} \otimes \ldots \otimes \tilde{e}_{a_{p+q}}$, the system of real numbers $\tilde{t}_{a_1a_2\ldots a_p}{}^{a_{p+1}\ldots a_{p+q}}$ satisfies

$$t_{i_1i_2\ldots i_p}{}^{i_{p+1}\ldots i_{p+q}} = b^a_1 b^a_2 \ldots b^a_p a_{a_{p+1}} \ldots a_{a_{p+q}} \tilde{t}_{a_1a_2\ldots a_p}{}^{a_{p+1}\ldots a_{p+q}}.$$

So we have

$$t_{i_1i_2\ldots i_p}{}^{i_{p+1}\ldots i_{p+q}} \theta^{i_1} \otimes \theta^{i_2} \otimes \ldots \otimes \theta^{i_p} \otimes e_{i_{p+1}} \otimes \ldots \otimes e_{i_{p+q}}$$

$$= \tilde{t}_{a_1a_2\ldots a_p}{}^{a_{p+1}\ldots a_{p+q}} \tilde{\theta}^{a_1} \otimes \tilde{\theta}^{a_2} \otimes \ldots \otimes \tilde{\theta}^{a_p} \otimes \tilde{e}_{a_{p+1}} \otimes \ldots \otimes \tilde{e}_{a_{p+q}}.$$

Often it would require long computations to verify the above equalities. Fortunately there are tensoriality criteria.

**0.18. Tensoriality Criteria.** Let $\{\omega_1\}_{1 \leq i \leq n}$ be a system $\omega$ of $n$ real numbers attached to the basis $\{e_i\}$ of $E$, and $\{\tilde{\omega}_\alpha\}_{1 \leq \alpha \leq n}$ the system $\omega$ in the basis $\{\tilde{e}_\alpha\}_{1 \leq \alpha \leq n}$. Suppose that for any vector $X \in E$, $\omega_i X^i = \tilde{\omega}_\alpha \tilde{X}^\alpha$. Then $\omega$ is a 1-form.

Since $X^i = a^i_\alpha \tilde{X}^\alpha$ (Example 0.15), we find that $(\omega_i a^i_\alpha - \tilde{\omega}_\alpha) \tilde{X}^\alpha = 0$ for any $X$. Thus $\tilde{\omega}_\alpha = \omega_i a^i_\alpha$. Multiplying the equality by $b^\alpha_i$, we get, after summation, $\omega_j = b^\alpha_j \tilde{\omega}_\alpha$. So $\omega$ is a $(1,0)$-tensor, that is a 1-form, according to Theorem 0.17. This result can be generalized.

**0.19. Theorem.** Suppose that a system $T$ of $n^{p+q}$ real numbers $t_{i_1i_2\ldots i_p}{}^{i_{p+1}\ldots i_{p+q}}$ in the basis $\{e_i\}$, $\tilde{t}_{a_1a_2\ldots a_p}{}^{a_{p+1}\ldots a_{p+q}}$ in the basis $\{\tilde{e}_\alpha\}$ satisfies the following condition: for any vectors $X_1, \ldots, X_p$ and any 1-forms
\[ \omega^1, \ldots, \omega^q, \]
\[ t_{i_1 i_2 \ldots i_p} t_{p+1 \ldots p+q} X_1^{i_1} X_2^{i_2} \ldots X_p^{i_p} \omega_{p+1}^{i_{p+1}} \ldots \omega_{p+q}^{i_{p+q}} = \tilde{t}_{\alpha_1 \ldots \alpha_p \alpha_{p+1} \ldots \alpha_{p+q}} \dot{X}_1^{\alpha_1} \ldots \dot{X}_p^{\alpha_p} \dot{\omega}_1^{\alpha_{p+1}} \ldots \dot{\omega}_p^{\alpha_{p+q}}. \]

Then \( T \) is a \( (p, q) \)-tensor.

**Proof.** For \( 1 \leq j \leq p \) we have \( X_j^i = a_j^i \dot{X}_j^\alpha \), and for \( 1 \leq k \leq q \) we have \( \omega_k^i = b_k^i \dot{\omega}_k^\alpha \). Putting these expressions in the condition of Theorem 0.19, we get
\[ t_{i_1 i_2 \ldots i_p} t_{p+1 \ldots p+q} a_{i_1}^{\alpha_1} \ldots a_{i_p}^{\alpha_p} b_{p+1}^{\alpha_{p+1}} \ldots b_{p+q}^{\alpha_{p+q}} = \tilde{t}_{\alpha_1 \ldots \alpha_p \alpha_{p+1} \ldots \alpha_{p+q}}. \]
Thus \( T \) is a \( (p, q) \)-tensor.

There are others tensoriality criteria.

**0.20. Example.** Suppose that a system \( T \) of \( n^3 \) real numbers \( t_{ij}^k \) in the basis \( \{ e_i \} \), \( \tilde{t}_{ij}^k \) in the basis \( \{ \tilde{e}_i \} \) satisfies the following condition: for any vectors \( X \) and \( Y \), \( t_{ij}^{k} X^i Y^j = Z^k \) are the components of a vector \( Z \); that is, we also have \( Z^\gamma = \tilde{t}_{ij}^k \tilde{X}^\alpha \tilde{Y}^\beta \). Then \( T \) is a \( (2,1) \)-tensor.

**Proof.** We have \( X^i = a_i^\alpha \dot{X}^\alpha \), \( Y^j = a_j^\beta \dot{Y}^\beta \) and \( Z^k = a_k^\gamma \dot{Z}^\gamma \). Thus \( \tilde{t}_{ij}^k a_i^\alpha a_j^\beta = a_k^\gamma \tilde{t}_{ij}^k \). Multiplying both sides by \( b_k^\lambda \), we get, after summation, \( t_{ij}^k a_i^\alpha a_j^\beta b_k^\lambda = \tilde{t}_{ij}^k a_k^\gamma \delta_{ij}^\lambda \). So \( T \) is a \( (2,1) \)-tensor.

**0.21. Definition.** The exterior product of two vectors \( x \) and \( y \) of \( E \) is the skew-symmetric tensor
\[ x \wedge y = x \otimes y - y \otimes x. \]
If \( \{ x^i \} \) and \( \{ y^j \} \) are the components of \( x \) and \( y \) in the basis \( \{ e_i \} \), then \( x^i y^j - x^j y^i = z^{ij} \) are the components of \( x \wedge y \) in the basis \( \{ e_i \wedge e_j \} \), \( 1 \leq i < j \leq n \), which has \( C_n^2 \) elements.

In this book we will consider especially the skew-symmetric \( p \)-forms on \( \mathbb{R}^n \) (or on a vector space \( E \) of dimension \( n \)), that are the skew-symmetric \( (p, 0) \)-tensors. If \( \{ x^i \}_1 \leq i \leq n \) are the coordinates on \( \mathbb{R}^n \) (or \( E \)) corresponding to a basis \( \{ e_i \}_1 \leq i \leq n \), we denote by \( dx^i \) the 1-form such that \( dx^i(e_j) = \delta_j^i \), the Kronecker tensor. A 1-form is defined when we know what it gives on the vectors of a basis, so \( dx^i \) is well defined. We denote by \( \Lambda^p E \) the space of the skew-symmetric \( p \)-forms on \( E \). More generally, the elements \( dx^{\lambda_1} \wedge dx^{\lambda_2} \wedge \cdots \wedge dx^{\lambda_p} \) \( 1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_p \leq n \) form a basis of \( \Lambda^p E \).

These forms are well defined when we know what they give on a system of \( p \) vectors of a basis. By definition,
\[ dx^{\lambda_1} \wedge dx^{\lambda_2} \wedge \cdots \wedge dx^{\lambda_p} (e_{\mu_1}, e_{\mu_2}, \ldots, e_{\mu_p}) = \delta_{\mu_1}^{\lambda_1} \delta_{\mu_2}^{\lambda_2} \cdots \delta_{\mu_p}^{\lambda_p}, \]
here $1 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_p \leq n$ and $1 \leq \mu_1 < \mu_2 < \cdots < \mu_p \leq n$. Indeed, we arrange the $p$ vectors in order. When we interchange two of them, the result changes sign.

0.22. Contraction of Indices. Let $T$ be a $(p, q)$-tensor. The contraction consists of choosing two indices, one covariant and the other contravariant. We set them equal and make summations, and thus get a $(p-1, q-1)$-tensor which is a tensor obtained from $T$ by contraction.

Let $t_{i_1, i_2, \ldots, i_p}^{j_{p+1} \cdots j_{p+q}}$ be the components of $T$ (see 0.17 for the notations). Choose for instance $i_1$ and $i_{p+1}$. We obtain, after summation over $j = i_1 = i_{p+1}$,

$$t_{i_2 \cdots i_p}^{j_{p+2} \cdots j_{p+q}} = t_{j, i_2, \ldots, i_p}^{j, i_{p+2} \cdots i_{p+q}},$$

which are the components of a $(p-1, q-1)$-tensor. Indeed, since $a_{p}^{i} b_{i_{p+1}}^{\beta} = \delta_{i_{p+1}}^{i}$, we have

$$\tilde{t}_{a_{p}^{\alpha} \alpha_{p+2} \cdots \alpha_{p+q}} = \tilde{t}_{\beta \alpha_{p+2} \cdots \alpha_{p+q}} = t_{i_1, i_2, \ldots, i_p}^{i_1, i_2 \cdots i_{p+q}} a_{p}^{i} b_{i_{p+1}}^{\beta} b_{i_{p+2}}^{\alpha_p+2} \cdots b_{i_{p+q}}^{\alpha_p+q} = t_{i_2 \cdots i_p}^{i_2 \cdots i_p+2 \cdots i_{p+q}} a_{i_2}^{\alpha_2} \cdots a_{i_p}^{\alpha_p} b_{i_{p+2}}^{\alpha_p+2} \cdots b_{i_{p+q}}^{\alpha_p+q}.$$

Differential Calculus

0.23. Definition. Let $f$ be a continuous map of an open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^p$. We say that $f$ is differentiable at $x_0 \in \mathbb{R}^n$ if there exists a linear mapping $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ such that for any $y \in \mathbb{R}^n$ with $x_0 + y \in \Omega$ we have

$$f(x_0 + y) - f(x_0) = A(y) + \|y\|\omega(x_0, y),$$

where $\omega(x_0, y) \to 0$ as $y \to 0$.

We call $A$ the differential of $f$ at $x_0$. If $f$ is differentiable at every point $x \in \Omega$, we say that $f$ is differentiable on $\Omega$. The differential of $f$ at $x$ depends on $x$; we denote it by $f'(x)$. If $\Omega \ni x \to f'(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ is continuous, we say that $f$ is continuously differentiable on $\Omega$, or $f$ is $C^1$.

There are functions which are differentiable on $\Omega$ and which are not $C^1$. Nevertheless, in this book when we talk about a differentiable function $f$, we assume that $f$ is at least $C^1$. We will never consider differentiable functions which are not $C^1$.

0.24. Remark. We can define differentiable maps of a Banach space $B_1$ into a Banach space $B_2$. The definition is the same as above except that the linear mapping $A$ must be continuous (for finite dimensional Banach spaces, $A$ linear implies $A$ continuous). Recall that a Banach space is a complete normed vector space. Complete means that any Cauchy sequence converges.
0.25. Let $f$ be a differentiable map of $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^p$. $f$ is defined by means of a system $f^1, f^2, \cdots, f^p$ of $p$ differentiable functions on $\Omega$ (real-valued): the coordinates of $f(x)$ in $\mathbb{R}^p$ are $f^1(x), f^2(x), \cdots, f^p(x)$. The differential $f'(x)$ is represented by the Jacobian matrix $((\partial f^\alpha/\partial x^i))_x$ with $n$ columns and $p$ rows.

Here $\{x^i\}_{1 \leq i \leq n}$ are coordinates on $\mathbb{R}^n$ ($\{e_i\}$ is a basis of $\mathbb{R}^n$), and $\{y^\alpha\}_{1 \leq \alpha \leq p}$ are coordinates on $\mathbb{R}^p$. Indeed, set $A = ((a^\alpha_i))$. From Definition 0.23 we get

$$f^\alpha(x + te_i) - f^\alpha(x) = ta^\alpha_i + |t|\omega^\alpha(x, te_i),$$

which is defined for $t$ in a neighbourhood of 0, and $\omega^\alpha(x, te_i) \to 0$ when $t \to 0$. Thus $f^\alpha$ is differentiable and $a^\alpha_i = \partial f^\alpha / \partial x^i$.

$f$ is $C^1$ on $\Omega$ if and only if the partial derivatives $\partial f^\alpha / \partial x^i$ ($1 \leq \alpha \leq p$, $1 \leq i \leq n$) exist and are continuous on $\Omega$.

The rank of $f$ at $x$ is the rank of $((\partial f^\alpha / \partial x^i))_x$; it is the dimension of the vector subspace of $\mathbb{R}^p$, the image of $\mathbb{R}^n$ under $f'(x)$.

0.26. Definition. We say that $f$ is $C^2$ on $\Omega$ ($f \in C^2(\Omega)$) if the map

$$\Omega \ni x \to f'(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$$

is $C^1$ on $\Omega$.

By induction we define $C^r$-functions ($r \in \mathbb{N}$). $C^\infty$-functions are functions which are $C^r$-functions for all $r \in \mathbb{N}$; in this case each $f^\alpha$ has partial derivatives of any order. $C^\infty$ functions are analytic functions.

Let $g$ be a $C^1$-function of $\theta \subset \mathbb{R}^p$ into $\mathbb{R}^m$. Suppose $f(\Omega) \subset \theta$; then $g \circ f$ is a $C^1$-function of $\Omega$ into $\mathbb{R}^m$ and $(g \circ f)' = g' \circ f'$. We will write also $df$ for $f'$; thus $d(g \circ f) = dg \circ df$. If $f$ is a homeomorphism of $\Omega$ onto $f(\Omega) \subset \mathbb{R}^n$ ($n = p$), then $f$ is a diffeomorphism if $f$ and $f^{-1}$ are differentiable. In this case $(f')^{-1} = (f^{-1})'$, since $d(f^{-1} \circ f)$ is the identity map.

0.27. Mean Value Theorem. Let $a, b$ be two points in $\Omega \subset \mathbb{R}^n$ such that the line segment $[a, b] \subset \Omega$, and let $f$ be a $C^1$-function of $\Omega$ into $\mathbb{R}^p$. Then

$$\|f(b) - f(a)\| \leq M\|b - a\|,$$

where $M = \sup_{x \in [a, b]} \|f'(x)\|$.

The set $[a, b] = \{a + t(b - a) / t \in [0, 1]\}$ is compact and $x \to \|f'(x)\|$ is continuous; thus $M$ is finite. Here the norm $\|\|$ is the norm as operator: if $u \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$, then $\|u\| = \sup_{\|x\|=1} \|u(x)\|$.

Proof. Set $\Phi(t) = f(a + ty)$ with $y = b - a$. Then $\Phi$ is a $C^1$ map of $[0, 1]$ into $\mathbb{R}^p$. According to the chain rule mentioned in 0.26, $\Phi'(t) = f'(a + ty)\circ y$. 

Thus
\[ f(b) - f(a) = \Phi(1) - \Phi(0) = \int_0^1 \Phi'(t) dt = \int_0^1 f'(a + ty) \circ y \, dt \]
and
\[ \|f(b) - f(a)\| \leq \int_0^1 \|f'(a + ty)\| \|y\| \, dt \leq M \|y\|. \]

0.28. Proposition. The image of \( \mathbb{R}^n \) by a \( C^1 \)-mapping \( f \) into \( \mathbb{R}^p \) is of (Lebesgue) measure zero if \( n < p \). The same result holds for an open set \( \Omega \subset \mathbb{R}^n \).

We give this result because we will use it in Chapter 1.

Recall that a set \( A \subset \mathbb{R}^p \) is of measure zero (\( \mu(A) = 0 \)) if for an arbitrary \( \epsilon > 0 \) there exist balls \( B_\alpha (\alpha \in \mathbb{N}) \) such that \( A \subset \bigcup_{\alpha \in \mathbb{N}} B_\alpha \) and \( \sum_{\alpha \in \mathbb{N}} \mu(B_\alpha) < \epsilon \).

Thus a countable union of sets of measure zero has measure zero.

Proof of Proposition 0.28. Let \( K_r = \{ x \in \mathbb{R}^n \mid |x^i| \leq r \text{ for } 1 \leq i \leq n \} \) (\( \{x^i\} \) are the coordinates of \( x \)). We slice each side of the cube \( K_r \) into \( k \) segments of the same length. We obtain \( k^n \) little cubes, with sides of length \( 2r/k \). If we consider \( K_r \subset \mathbb{R}^n \subset \mathbb{R}^p \), \( K_r \) has measure zero in \( \mathbb{R}^p \).

Indeed, \( K_r \subset \bigcup_{j=1}^{k^n} B_{P_j}(\rho) \) with \( \rho = r\sqrt{n}/k \), where \( B_{P_j}(\rho) \) is the ball in \( \mathbb{R}^p \) of radius \( \rho = r\sqrt{n}/k \) and center \( P_j \), the center of a little cube. So \( \mu(K_r) \leq k^n \mu(B_0(\rho)) = k^n \mu(B_0(r\sqrt{n})) \), where \( \mu \) denotes the measure in \( \mathbb{R}^p \). When \( k \to \infty \), the right hand side goes to zero; then \( \mu(K_r) = 0 \).

Let \( f \) be a \( C^1 \)-mapping of \( \mathbb{R}^n \) into \( \mathbb{R}^p \). On \( K_r \) we have \( \|f'\| \leq M_r \), and by the mean value theorem, for \( x, y \) in \( K_r \), \( \|f(x) - f(y)\| \leq M_r \|x - y\| \). Thus the little cube \( K_j \) of center \( P_j \) is such that \( f(K_j) \subset B_{f(P_j)}(\rho M_r) \). This implies
\[ \mu(f(K_r)) \leq k^n \mu(B_0(\rho M_r)) = k^n \mu(B_0(r\sqrt{n})) \],
which goes to zero when \( k \to \infty \). So \( \mu(f(K_r)) = 0 \). As \( \mathbb{R}^n = \bigcup_{r=1}^{\infty} K_r \), we see that \( f(\mathbb{R}^n) = \bigcup_{r=1}^{\infty} f(K_r) \). According to the theorem cited above, \( \mu(f(\mathbb{R}^n)) = 0 \), since \( \mu(f(K_r)) = 0 \) for all \( r \in \mathbb{N} \). For general \( \Omega \), there exists a sequence of compact sets \( \{K_i\} \) such that \( K_i \subset K_{i+1} \) and \( \Omega = \bigcup_{i \in \mathbb{N}} K_i \). We prove that \( \mu(A(K_i)) = 0 \), using the result above. For this we only have to consider a \( C^1 \)-mapping \( \bar{f} \) of \( \mathbb{R}^n \) into \( \mathbb{R}^p \) such that \( \bar{f}/K_i = f/K_i \). Then \( \mu(f(K_i)) = 0 \) for all \( i \) implies \( \mu(f(\Omega)) = 0 \).

0.29. Inverse Function Theorem. Let \( f \) be a \( C^1 \)-mapping of an open set \( \Omega \subset \mathbb{R}^n \) into \( \mathbb{R}^n \). If \( f'(x_0) \) is invertible (rank \( f = n \) at \( x_0 \in \Omega \)), there exists an open neighbourhood \( \theta \) of \( x_0 \) such that \( \Phi = f|_\theta \) is a diffeomorphism of \( \theta \) onto \( f(\theta) \). If \( f \in C^k \), then \( \Phi \) and \( \Phi^{-1} \) are \( C^k \) on \( \theta \).
**Proof.** Let us consider the mapping \( g(x) = f^{-1}(x_0) \circ f(x) \). We have \( g'(x_0) = \text{Identity.} \) Set \( h(x) = g(x) - x \). \( h \) is \( C^1 \) on \( \Omega \), and since \( h'(x_0) = 0 \), there exists \( r > 0 \) such that \( \|x - x_0\| < r \) implies \( \|h'(x)\| < \frac{1}{2} \). We choose \( r \) small enough so that rank \( f = n \) on the ball \( \tilde{B} \subset \Omega \) of radius \( r \) and center \( x_0 \). According to the mean value theorem, for \( x \) and \( x' \) in \( \tilde{B} \)
\[
\|h(x) - h(x')\| < \frac{1}{2}\|x - x'\|.
\]
Let \( B \) be a ball of center \( y_0 = g(x_0) \) and radius smaller than \( r/2 \). Consider the equation \( x + h(x) = y \) for \( y \in B \) given. We define by induction the sequence \( x_{k+1} = y - h(x_k) \) \((k \geq 0)\). We have
\[
\|x_{k+1} - x_k\| = \|h(x_k) - h(x_{k-1})\| < \frac{1}{2}\|x_k - x_{k-1}\| < \frac{1}{2^k}\|x_1 - x_0\| = \frac{1}{2^k}\|y - y_0\|
\]
since \( y \in B \) implies \( x_k \in \tilde{B} \). Indeed, \( x_1 = y - [g(x_0) - x_0] \) and
\[
\|x_k - x_0\| \leq \sum_{l=1}^{k} \|x_l - x_{l-1}\| < \|y - y_0\| \sum_{l=1}^{k} \frac{1}{2^{l-1}} < 2\|y - y_0\|.
\]
The sequence \( \{x_k\} \) is a Cauchy sequence. So \( x_k \to x \in \tilde{B} \). Since \( h \) is continuous, \( x = y - h(x) \). So \( y = g(x) \). Moreover, the solution is unique in \( \tilde{B} \), since \( \|h'(z)\| < \frac{1}{2} \) for \( z \in \tilde{B} \). Indeed, \( \|x - \bar{x}\| = \|h(x) - h(\bar{x})\| \leq \frac{1}{2}\|x - \bar{x}\| \) implies \( x = \bar{x} \).

Set \( \Theta = \tilde{B} \cap g^{-1}(B) \); then \( \Phi = f|_{\Theta} \) is bijective. Let us show that \( \Phi^{-1} \) is differentiable on \( f(\Theta) \). For all \( \epsilon > 0 \), there is \( \eta > 0 \) such that \( \|x - x_0\| < \eta \) implies
\[
\|g(x) - g(x_0) - x + x_0\| < \epsilon\|x - x_0\|,
\]
since \( h'(x_0) = 0 \). Thus
\[
\|f'^{-1}(x_0) \circ (y - y_0) - \Phi^{-1}(y) + \Phi^{-1}(y_0)\| < 2\epsilon\|y - y_0\|.
\]
This inequality proves that \( \Phi^{-1} \) is differentiable at \( x_0 \). Its differential is \( f'^{-1}(x_0) \), and rank \( \Phi^{-1} = n \) at \( y_0 \). As we can give the same proof at any point \( \bar{x}_0 \in \Theta \), it follows that \( \Phi^{-1} \) is differentiable on \( f(\Theta) \). Since \( (\Phi^{-1})'(y) = f'^{-1}[\Phi^{-1}(y)] \), it follows that \( \Phi^{-1} \) is \( C^1 \) on \( f(\Theta) \). \( \Phi^{-1} \) is \( C^k \) if \( f \) is \( C^k \)

**0.30. Remark.** The inverse function theorem holds in Banach spaces. The proof is the same.

And now a global result.

**0.31. Theorem.** Let \( f \) be a \( C^1 \)-mapping of \( \Omega \subset B \) into \( \tilde{B} \), \( B \) and \( \tilde{B} \) being two Banach spaces. Suppose that

a) \( f \) is injective, and

b) \( f'(x) \) is an isomorphism from \( B \) onto \( \tilde{B} \) for every \( x \in \Omega \).
Then $f$ is a diffeomorphism from $\Omega$ onto $f(\Omega) \subset B$.

Since $f$ is injective, $\Phi = f|_{\Omega} : \Omega \to f(\Omega)$ is invertible; i.e., $\Phi^{-1}$ exists. According to 0.29, $\Phi^{-1}$ is continuous and $C^1$-differentiable (these are local properties).

0.32. Let $B$ be a Banach space, and let $(t, x) \to f(t, x) \in B$ be a continuous function defined in a neighbourhood of $(t_0, x_0)$ in $\mathbb{R} \times B$. Since $f$ is continuous, there exists a neighbourhood $V$ of $(t_0, x_0)$ where $f(t, x)$ is bounded:

$$\|f(t, x)\| \leq M \text{ for all } (t, x) \in V.$$

We consider a closed ball $\Omega$ of radius $r$ and center $x_0$ in $B$, and $I = [t_0 - \alpha, t_0 + \alpha] \subset \mathbb{R}$ ($\alpha > 0$) such that $I \times \Omega \subset V$, $\alpha$ and $r$ satisfying $M\alpha < r$. Recall that a map $h$ of an open set $\Theta$ of a Banach space $B_1$ into a Banach space $B_2$ is Lipschitz in $\Theta$ if there exists $k \in \mathbb{R}$ such that $\|h(a) - h(b)\| \leq k\|a - b\|$ for $(a, b) \in \Theta \times \Theta$.

0.33. The Cauchy Theorem. The differential equation

$$(*) \quad x' = f(t, x), \quad x(t_0) = x_0,$$

has a unique continuous solution $x(t)$ if $f(t, x)$ is Lipschitz in $x$ on $I \times \Omega$. The solution is defined on a neighbourhood $J \subset I$ of $t_0$, and its values are in $\Omega$.

Proof. First of all, a continuous function $x(t)$ satisfying $(*)$ satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(u, x(u)) \, du,$$

and conversely. Let $C(I, B)$ be the Banach space of all continuous functions $g$ on $I$ into $B$ endowed with the norm $\sup_{t \in I} \|g\|$. Consider the mapping $\Phi$ of $C(I, \Omega)$ into itself defined by

$$\Phi : C(I, \Omega) \ni x(t) \to y(t) = x_0 + \int_{t_0}^{t} f(u, x(u)) \, du.$$

Since $f$ is Lipschitz in $x$, there exists $k$ such that, for $t \in I$ and $a, b$ in $\Omega$,

$$\|f(t, a) - f(t, b)\| \leq k\|a - b\|.$$

Thus $\Phi$ is locally a contracting mapping. Indeed,

$$\|\Phi(x) - \Phi(y)\| \leq |t - t_0|\|f(t, x) - f(t, y)\| \leq k|t - t_0|\|x(t) - y(t)\|.$$

Pick $\beta$ ($0 < \beta < \alpha$) such that $\beta k < 1$, and set $J = [t_0 - \beta, t_0 + \beta] \subset I$. Then $\Phi$ is a contracting mapping of the Banach space $C(J, B)$ into itself. The fixed point theorem then implies the existence of a unique $z \in C(J, B)$ such that $\Phi(z) = z$. 


0.34. **Remark.** If \( f(t, x) \) is not Lipschitz, we cannot say anything in general. But if the Banach space is \( \mathbb{R}^n \), then there is a solution of (\( * \)), but it may not be unique. In that case there exist a solution greater than the others, and a solution smaller than the others.

0.35. **Example.** \( n = 1, x' = 2\sqrt{|x|}, x(0) = 0. \)

0.36. **Dependence on Initial Conditions.** The solution of (\( * \)) (Theorem 0.33) depends on the initial conditions \((t_0, x_0)\). Thus we can write it in the form \( x(t, t_0, x_0) \).

If we choose \((\hat{t}, \hat{x})\) in a neighbourhood \( \theta \) of \((t_0, x_0)\), we can choose \( r \) and \( \alpha \) small enough so that the closed ball \( \Omega_{\hat{x}} \) of radius \( r \) and center \( \hat{x} \), and \( \tilde{I} = [\tilde{t} - \alpha, \tilde{t} + \alpha] \), satisfy \( \tilde{I} \times \Omega_{\hat{x}} \subset V \) for any \((\tilde{t}, \hat{x}) \in \theta \).

0.37. **Theorem.** Let \( f \) be a continuous function, Lipschitz in \( x \), as in Theorem 0.33. Then there exists a neighbourhood \( \theta \) of \((t_0, x_0)\) in \( \mathbb{R} \times B \) such that the solution \( x(t, \hat{t}, \hat{x}) \) of the differential equation

\[
x' = f(t, x), \quad x(\hat{t}) = \hat{x},
\]

exists on \([\tilde{t} - \beta, \tilde{t} + \beta], \beta > 0 \) being independent of \((\hat{t}, \hat{x})\). Moreover, \((t, \hat{t}, \hat{x}) \rightarrow x(t, \hat{t}, \hat{x}) \) is continuous. If \( f \) is \( C^\infty \), this map is \( C^\infty \).
Exercises and Problems

0.38. Problem. Let \( t \to x(t) \in \mathbb{R} \) be a function defined on an interval of \( \mathbb{R} \). Consider the family of first order differential equations \( E_\lambda (\lambda \in \mathbb{R}) \):

\[
x' = x^2(1 + t^2\lambda^2) - 1, \quad x(0) = 2.
\]

a) Show that \( E_\lambda \) has a unique maximal \( C^\infty \) solution \( x_\lambda \) defined on an interval \( I_\lambda = (a_\lambda, b_\lambda) \).

b) Integrate the equation \( E_0 \) explicitly.

c) Let \( f(t, x) \) and \( g(t, x) \) be two continuous functions on \( \mathbb{R}^2 \) with value in \( \mathbb{R} \), and suppose that \( f \) and \( g \) are uniformly Lipschitz in \( x \). Let \( (t_0, x_0) \) be initial conditions, \( y(t) \) the maximal solution of the equation

\[
x' = f(t, x), \quad x(t_0) = x_0
\]

(it exists on \( I \)), and \( z(t) \) the maximal solution of the equation

\[
x' = g(t, x), \quad x(t_0) = x_0
\]

(it exists on \( J \)). If \( f(t, x) < g(t, x) \) on \( \mathbb{R}^2 \), prove that \( z(t) > y(t) \) for \( t_0 < t \in I \cap J \) and that \( z(t) < y(t) \) for \( t_0 > t \in I \cap J \).

d) If \( f(t, x) \leq g(t, x) \) on \( \mathbb{R}^2 \), prove that \( z(t) \geq y(t) \) for \( t_0 \leq t \in I \cap J \) and that \( z(t) \leq y(t) \) for \( t_0 \geq t \in I \cap J \). Hint. Consider the family of functions \( g_n(t, x) = g(t, x) + \frac{1}{n} \) with \( n \in \mathbb{N} \).

e) Show that \( b_\lambda \) is finite. What is the upper bound of \( b_\lambda \)?

f) Prove that \( x_\lambda > 0 \) on \( I_\lambda \). Hint. Argue by contradiction, considering the equation \( x' = x^2(1 + t^2\lambda^2) \).

g) Establish that \( x_\lambda(t) \leq 2 \) for \( t \leq 0 \). Deduce from the previous question that \( a_\lambda = -\infty \).

0.39. Problem. Let \( f(t, x) \) be a continuous map of \( I \times \Omega \subset \mathbb{R} \times \mathbb{R}^n \) into \( \mathbb{R}^n \), where \( I = [t_0, t_0 + \alpha] \) and \( \Omega = \overline{B}_{x_0}(r) \subset \mathbb{R}^n \) (\( \alpha > 0, r > 0, x_0 \in \mathbb{R}^n, B_{x_0}(r) \) the ball with center \( x_0 \) and radius \( r \)). Set \( M = \sup_{(t, x) \in I \times \Omega} ||f(t, x)|| \) for \( (t, x) \in I \times \Omega \), and choose \( \alpha \leq \frac{r}{M+1} \).

a) Consider for \( t \in [t_0 - \alpha, t_0] \) the function

\[
y_0(t) = x_0 + f(t_0, x_0)(t - t_0)
\]

and the functions \( y_k (\frac{1}{\alpha} \leq k \in \mathbb{N}) \) defined by \( y_k(t) = y_0(t) \) for \( t \in [t_0 - \alpha, t_0] \) and \( y_k(t) = x_0 + \int_{t_0}^{t} f(s, y_k(s - \frac{1}{k}))\,ds \) for \( t > t_0 \). Show that \( y_k(t) \) is defined and continuous on \( [t_0 - \alpha, t_0 + \frac{k}{\alpha}] \), and then on \( [t_0 - \alpha, t_0 + \alpha] \).

b) Prove that the family \( y_k(t) \) is equicontinuous on \( I \)—that is to say, for any \( \epsilon > 0 \), there exists \( \eta > 0 \) such that \( |t-s| \leq \eta \Rightarrow ||y_k(t) - y_k(s)|| < \epsilon \) for all \( k \geq \frac{1}{\alpha} \), \( t \) and \( s \) belonging to \( I \). Then apply Ascoli's theorem:
there exists a subsequence \( \{ y_{k_i} \} \subset \{ y_k \} \) which converges uniformly on \( I \) to a function \( z \). Show that \( z \) satisfies on \( I \) the equation \( E \):

\[
z' = f(t, z), \quad z(t_0) = x_0.
\]

c) From now on \( n = 1 \) (\( y \in \mathbb{R} \) and \( f(t, y) \in \mathbb{R} \)). Consider the equation \( E_p \) (\( p \in \mathbb{N} \)):

\[
y' = f(t, y) + \frac{1}{p}, \quad y(t_0) = x_0.
\]

Prove that \( E_p \) has at least one \( C^1 \)-solution defined on \( I \). Let \( p < q \) be two integers, \( z_p \) a solution of \( E_p \) and \( z_q \) a solution of \( E_q \). Prove that \( z_p(t) \geq z_q(t) \) for \( t \in I \). Deduce that \( \tilde{z}(t) = \lim_{p \to \infty} z_p(t) \) is a solution of \( E \) larger than any other solution of \( E \). \( \tilde{z} \) is called the maximal solution of \( E \).

0.40. Problem. Let \( t \to x(t) \in \mathbb{R} \) be a function defined on an open set of \( \mathbb{R} \). Consider the differential equation

\[
(E) \quad x' = \lambda - f(t)e^x,
\]

where \( f \) is a nonvanishing continuous periodic function on \( \mathbb{R} \) with period 1, and \( \lambda \) is a real parameter.

a) For \( t_0 \in \mathbb{R} \), show that there exists, in a neighbourhood of \( t_0 \), a unique differentiable solution of \( (E) \) such that \( x(t_0) = x_0 \), \( x_0 \) being a given real number.

b) Verify that if \( x \) is a solution of \( (E) \), then \( y = e^{-x} \) is a solution of

\[
(*) \quad y' + \lambda y = f.
\]

c) When \( \lambda \neq 0 \), prove that

\[
y_\lambda(t) = \frac{1}{e^\lambda - 1} \int_0^1 e^{\lambda u} f(t + u) \, du
\]

is the unique differentiable solution of \( (\ast) \) which is periodic of period 1.

d) What can we say about the existence of a periodic solution of \( (\ast) \) in the case \( \lambda = 0 \)?

e) When \( f \geq 0 \), deduce from c) that \( (E) \) has a periodic solution (of period 1) if and only if \( \lambda > 0 \).

f) Show that \( \lim_{\lambda \to -\infty} \lambda y_\lambda(t) = f(t) \). Deduce that \( (E) \) has a periodic solution (of period 1) for all \( \lambda < 0 \) if and only if \( f(t) \leq 0 \) for all \( t \).

g) If \( \int_0^1 f(t) \, dt < 0 \), establish the existence of \( \epsilon > 0 \) such that \( (E) \) has a periodic solution (of period 1) for any \( \lambda \in (-\epsilon, 0) \).
0.41. Problem. Let 
\[ E = \{ f \in C^1([0,1]) \mid f(0) = 0 \text{ and } f(1) = 1 \}. \]
What is the greatest real number \( m \) such that 
\[ m \leq \int_0^1 |f'(x) - f(x)| \, dx \]
for any \( f \in E \)? *Hint.* Consider the function \( fe^{-x} \).

0.42. Problem. Let \( t \to x(t) \in \mathbb{R} \) be a function defined on an open set of \( \mathbb{R} \). Consider the differential equation
\[ (E_\lambda) \quad x' = \lambda + \frac{x^2}{1 + t^2}, \quad x(0) = 0, \quad \lambda \in \mathbb{R}. \]

a) Prove that there exists \( \lambda_0 \) such that \( (E_\lambda) \) has a solution on \([0, \infty)\) if and only if \( \lambda \leq \lambda_0 \). What is the value of \( \lambda_0 \)?

b) When \( \lambda > \lambda_0 \), the maximal solution exists on \([0, a_\lambda)\). Show that
\[ \sinh \frac{\pi}{2\sqrt{\lambda}} < a_\lambda < \sinh \frac{\pi}{\sqrt{\lambda - \frac{1}{4}}}. \]

0.43. Exercise. Let \( f \) be a real valued differentiable function at each point of \([a,b] \subset \mathbb{R}\), and suppose that \( f'(a) < f'(b) \). Let \( y_0 \in (f'(a), f'(b)) \). Prove that there exists \( x_0 \in (a,b) \) such that \( f'(x_0) = y_0 \).

0.44. Problem. Let \( \lambda_n \) (\( n \in \mathbb{N} \)) be the positive solutions of \( \tan x = x \).

a) Show that
\[ \sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \frac{1}{10}. \]

b) Consider the equation
\[ (E) \quad -y'' + b^2 y = f(x), \quad y(0) = 0, \quad y'(1) = y(1), \]
with \( b > 0 \) and \( f \) a continuous function. Is there a solution? Is it unique?

c) Find a function \( G(x, t) \) on \([0,1] \times [0,1]\) such that the solution of \((E)\) is
\[ y(x) = \int_0^1 G(x, t)f(t) \, dt. \]

d) Is \( G(x, t) \) continuous? Does it satisfy \( G(x, t) = G(t, x) \)?

e) If the equation
\[ -y'' + b^2 y = \mu y, \quad y(0) = 0, \quad y'(1) = y(1), \]
has a non-trivial solution, what can we say about \( \mu \)?
0.45. Exercise. Consider the differential equation
\[ x' = x^2 + t, \quad x(0) = 0, \]
where \( t \to x(t) \in \mathbb{R} \) is a function on a neighbourhood of \( 0 \in \mathbb{R} \).

a) Show that a solution exists on \((-b, a)\) with \( a < 3 \).

b) What can we say about \( b \)?

0.46. Exercise. Consider the differential equation \( y' = x - \frac{1}{y} \). Then \( x \rightarrow y(x) \in \mathbb{R} \) is a function defined on an interval of \( \mathbb{R} \). Prove that it has a unique solution on \([0, \infty)\) which is positive and which tends to zero when \( x \rightarrow \infty \).

0.47. Problem. In this problem the given functions and the solutions are defined on \( \mathbb{R} \), with values in \( \mathbb{R} \). They are even and periodic of period \( 2\pi \). \( E_0 \) will be the space of continuous functions on \( \mathbb{R} \) which are even and periodic of period \( 2\pi \). For \( k \in \mathbb{N} \), we have \( E_k = \{ f \in E_0 \mid f \in C^k \} \). Let \( C^0_B \) be the space of the bounded continuous functions on \( \mathbb{R} \) endowed with the norm \( ||f||_{C^0} = \sup |f| \), and \( P_n(x) \) the set of the functions in \( E_1 \) whose Fourier series have vanishing coefficients \( a_k \) when \( k > n \) (\( P_n(x) \) is a linear combination of the functions \( \cos kx \) with \( 0 \leq k \leq n \)).

Part I

a) Let \( h \in E_0 \). Show that for the equation \( y'' + y' \cotan x = h(x) \) to have a solution in \( E_2 \) on the open set \( \Omega \subset \mathbb{R} \), where \( \tan x \neq 0 \), it is necessary and sufficient that \( \int_0^\pi h(x) \sin x \, dx \). What can we say about uniqueness?

b) For \( \mu \) a real number, verify that
\[ \Gamma : y \to y'' + y' \cotan x - \mu y \]
is a map in \( P_n(x) \).

Let \( f_n \in P_n(x) \), and prove that if \( \mu > 0 \) (which is assumed henceforth), then the equation
\[ y'' + y' \cotan x - \mu y = f_n(x) \]
has a unique solution in \( E_2 \) (that means that the function in \( E_2 \) satisfies (1) on \( \Omega \)).

c) Let \( f \in E_1 \), and denote by \( f_n \) the partial sum up to order \( n \) of the Fourier series of \( f \). For each \( n \in \mathbb{N} \), we consider the solution \( y_n \) in \( E_2 \) of equation (1). Given \( k \) points \( x_1, x_2, \ldots, x_k \) of \([0, \pi] \), show that there exists a subsequence \( \{y_{p}\} \subset \{y_n\} \) which converges at these \( k \) points.

d) Prove that there exists a subsequence of \( \{y_n\} \) which is a Cauchy sequence in \( C^0_B \). Deduce that the equation
\[ y'' + y' \cotan x - \mu y = f(x) \]
has a unique solution in $E_2$.

e) Since $f \in E_1$, what is the regularity of the solution?

Part II

We next study the equation

$$z'' + z' \cotan x + \tilde{h}(x) = \tilde{f}(x)e^{\nu z},$$

where $\tilde{h}$, $\tilde{f}$ belong to $E_0$ and $\nu \in \mathbb{R}$, $\tilde{f} \neq 0$ and $\nu \neq 0$.

a) Reduce the study of (3) to the study of the equation

$$y'' + y' \cotan x + a = f(x)e^y,$$

where $a$ is constant and $f \in E_0$. When $f$ has a constant sign, establish that for (4) to have a solution in $R_2$, it is necessary that $a$ have the sign of $f$.

When $a = 0$, verify that for (4) to have a solution in $E_2$, it is necessary that $f$ changes sign and that $\int_0^\pi f(x) \sin x \, dx > 0$.

b) For the rest of Part II we suppose $a > 0$ and $f(x) > 0$ for all $x \in \mathbb{R}$. Exhibit two real numbers $m$ and $M$ such that $f(x)e^m < a < f(x)e^M$ for all $x \in \mathbb{R}$. Then consider the sequence of functions defined by induction as follows: $\varphi_0 \equiv m$ and, for $k > 0$, $\varphi_k$ is the solution in $E_2$ of the equation

$$\varphi''_k + \varphi'_k \cotan x - \mu \varphi_k = f(x)e^{\varphi_{k-1}} - a - \mu \varphi_{k-1},$$

where $\mu > 0$ is a real number. Prove that $\varphi_1 > \varphi_0$.

c) If $\mu$ is chosen large enough, establish that the sequence $\{\varphi_k\}$ is increasing and bounded by $M$.

d) Prove that $\{\varphi_k\}$ is a Cauchy sequence in $C_B^0$.

Deduce that equation (4) has a unique solution in $E_2$. What is its regularity?

Part III

In this part we suppose $a < 0$.

a) Verify that for (4) to have a solution in $E_2$, it is necessary that $f$ be negative at least somewhere. When $f(x) \equiv -2$ and $a = -2$, the equation

$$y'' + y' \cotan x + 2e^y = 2$$

has an obvious solution $y_0$. But in fact there exists a one-parameter family of solutions $y_t$ of (6) in $E_2$, $y_t$ being $C^1$ in a neighbourhood of $t$. Find the equation (7) satisfied by $w = (dy_t/dt)_{t=0}$.

b) Find the solutions of (7) in $E_2$. Let $\psi$ be one of them, $\psi \neq 0$. 
c) Find solutions of (6) in $E_2$ of the form

$$y = k \log(\mu(1 + \epsilon \psi)),$$

where $k$, $\mu$ and $\epsilon$ are real numbers to be chosen. Find all the solutions of (6) in $E_2$.

d) If equation (4) has one or more solutions in $E_2$, let $y$ be one of them. Prove the following identity:

$$\int_0^\pi f' \sin^2 x e^y \, dx = -(1 + a/2) \int_0^\pi f e^y \sin 2x \, dx.$$

When $a = -2$, show that equation (4) does not always have a solution, even if the necessary condition found in III a) is satisfied.

Specialists will recognize the Kazdan-Warner condition for the so-called Nirenberg problem (see Aubin [2]).
One begins a new field in mathematics with some definitions, and this course is no exception. There are many definitions, especially at the beginning. The subject of our study is differentiable manifolds. It is necessary to understand well what a differentiable manifold is.

We give the proof of the theorem on partition of unity, very useful in differential geometry. This proof needs point-set topology. The reader is assumed to know the definition of a topology and that of a compact set (see Chapter 0). But what is useful throughout the book is differential calculus. One must know what a differentiable mapping is, and the Cauchy Theorem on ordinary differential equations.

This chapter continues with the definition of a submanifold. To prove that a subset of a manifold is a submanifold, using the definition, seems to be difficult; fortunately we have at our disposal Theorem 1.19, which will be very useful for applications.

We end the chapter with two basic theorems, Whitney’s and Sard’s. We give the difficult proof of Whitney’s theorem, because it is a beautiful application of the knowledge already acquired. The reader may skip the proofs of Whitney’s theorem and the theorem on partition of unity.

Basic Definitions

1.1. Definition. A manifold $M_n$ of dimension $n$ is a Hausdorff topological space such that each point $P$ of $M_n$ has a neighbourhood $\Omega$ homeomorphic to $\mathbb{R}^n$ (or equivalently to an open set of $\mathbb{R}^n$).
More generally, we define a Banach manifold: each point has a neighbourhood homeomorphic to an open set of a Banach space. Here we will study only manifolds of finite dimension.

The notion of dimension makes sense because there is no homeomorphism of \( \mathbb{R}^n \) into \( \mathbb{R}^p \) if \( n \neq p \). We do not prove this main result, because we will study differentiable manifolds, for which the proof of the notion of dimension is obvious. We will only consider connected manifolds. If a manifold has more than one component, we study one component at a time.

1.2. Proposition. A manifold is locally compact and locally path connected.

By definition, locally compact (resp. locally path connected) means that every point has a basis of compact (resp. path connected) neighbourhoods (a family of neighbourhoods is a basis of neighbourhoods at \( P \), if any neighbourhood of \( P \) contains a neighbourhood of the family). A set \( E \) is path connected if, given any pair of points \( P, Q \) in \( E \), there is an arc in \( E \) from \( P \) to \( Q \).

An arc of \( M_n \) is the image, by a continuous map, of \([0, 1] \subset \mathbb{R}\) into \( M_n \). If \( \Gamma \) is an arc in \( \mathbb{R}^n \) and \( \varphi \) the homeomorphism of Definition 1.1, \( \varphi^{-1}(\Gamma) \) is an arc in \( M_n \). If there exist an arc from \( P \) to \( Q \) and another from \( Q \) to \( T \), their union is an arc from \( P \) to \( T \).

Let \( P \in M_n \) and let \( \Omega \) be a neighbourhood of \( P \) homeomorphic to an open set of \( \mathbb{R}^n \), \( \varphi : \Omega \to \mathbb{R}^n \). In this chapter, \( B_r \) will be the ball of \( \mathbb{R}^n \) with center \( O \) and radius \( r \). As often, we suppose without loss of generality that \( \varphi(P) = 0 \). \( \varphi^{-1}(B_r) \), \( r = 1, \frac{1}{2}, \ldots, \frac{1}{p}, \ldots \) (\( p \in \mathbb{N} \)), form a basis of
neighbourhoods of $P$ which are compact and path connected.

1.3. Proposition. A connected manifold is path connected.

Proof. Let $P \in M_n$, and let $W$ be the set of points $Q$ of $M_n$ for which there is an arc from $P$ to $Q$. $W$ is closed. Indeed, let $T \in \overline{W}$, and $U$ a neighbourhood of $T$ homeomorphic to $\mathbb{R}^n$. We have $U \cap W \neq \emptyset$; thus there exist $Q \in U \cap W$ and an arc from $P$ to $Q$; there is also an arc from $Q$ to $T$. Hence $T \in W$.

$W$ is open. Indeed, let $Q \in W$; $Q$ has a neighbourhood $\Omega$ homeomorphic to $\mathbb{R}^n$ and $\Omega \subset W$. Since $M_n$ is connected and $W \neq \emptyset (P \in W)$, $W = M_n$.

1.4. Definition. A local chart on $M_n$ is a pair $(\Omega, \varphi)$, where $\Omega$ is an open set of $M_n$ and $\varphi$ a homeomorphism of $\Omega$ onto an open set of $\mathbb{R}^n$. A collection $(\Omega_i, \varphi_i)_{i \in I}$ of local charts such that $\bigcup_{i \in I} \Omega_i = M_n$ is called an atlas. The coordinates of $P \in \Omega$ related to the local chart $(\Omega, \varphi)$ are the coordinates of the point $\varphi(P)$ in $\mathbb{R}^n$.

1.5. Definition. An atlas of class $C^k$ (respectively $C^\infty$, $C^\omega$), on $M_n$ is an atlas for which all changes of charts are $C^k$ (respectively $C^\infty$, $C^\omega$). That is to say, if $(\Omega_\alpha, \varphi_\alpha)$, and $(\Omega_\beta, \varphi_\beta)$ are two local charts with $\Omega_\alpha \cap \Omega_\beta \neq \emptyset$, then
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the map $\varphi_\alpha \circ \varphi_\beta^{-1}$, called change of charts, of $\varphi_\beta(\Omega_\alpha \cap \Omega_\beta)$ onto $\varphi_\alpha(\Omega_\alpha \cap \Omega_\beta)$ is a diffeomorphism of class $C^k$ (respectively $C^\infty, C^\omega$).

\[
\mathbb{R}^n \supset \varphi_\beta(\Omega_\alpha \cap \Omega_\beta) \xrightarrow{\varphi_\beta^{-1}} \Omega_\alpha \cap \Omega_\beta \xrightarrow{\varphi_\alpha} \varphi_\alpha(\Omega_\alpha \cap \Omega_\beta) \subset \mathbb{R}^n
\]

We consider the following relation of equivalence between atlases of class $C^k$ on $M_n$: two atlases $(U_i, \varphi_i)_{i \in I}$ and $(W_\alpha, \Psi_\alpha)_{\alpha \in \Lambda}$ of class $C^k$ are said to be equivalent if their union is an atlas of class $C^k$. That is to say that $\varphi_i \circ \Psi_\alpha^{-1}$ is $C^k$ on $\Psi_\alpha(U_i \cap W_\alpha)$ when $U_i \cap W_\alpha \neq \emptyset$.

1.6. Definition. A differentiable manifold of class $C^k$ (respectively, $C^\infty$ or $C^\omega$) is a manifold together with an equivalence class of $C^k$ atlases (respectively, $C^\infty$ or $C^\omega$).

On a manifold there need not always exist a differentiable atlas (of class $C^k$), but if there exists an atlas of class $C^1$, then there are atlases of class $C^\infty$ (which are $C^1$-equivalent to it) if the manifold is paracompact. It is possible now to talk about differentiable functions $C^k$ on a $C^p$-differentiable manifold when $k \leq p$. A function $f$ on $M_n$ (unless we say otherwise, a function takes its values in $\mathbb{R}$) is $C^k$-differentiable at $P \in M_n$ if for a local chart $(U, \varphi)$ with $P \in U$ the function $f \circ \varphi^{-1}$ is $C^k$-differentiable at $\varphi(P)$. We easily verify that this definition makes sense—the notion of differentiability does not depend on the local chart. Indeed, let $(\Omega, \psi)$ be another local chart at $P$; then $f \circ \psi^{-1} = f \circ \varphi^{-1} \circ \varphi \circ \psi^{-1}$ is $C^k$-differentiable at $\psi(P)$ since $\varphi \circ \psi^{-1}$
is \( C^k \)-differentiable because \( k \leq p \).

1.7. **Remark.** We can define complex manifolds \( M \). Consider an atlas of local charts \( (\Omega_i, \varphi_i)_{i \in I} \), where \( \varphi_i \) is a homeomorphism of \( \Omega_i \) onto an open set in \( \mathbb{C}^m \). If any change of charts \( \varphi_j \circ \varphi_i^{-1} \) is holomorphic on \( \varphi_i(\Omega_i \cap \Omega_j) \), \( M \) is a complex manifold of complex dimension \( m \) (\( n = 2m \)).

1.8. **Example.** An open set \( \Omega \) of a differentiable manifold \( M_n \) is a differentiable manifold. It is endowed with the atlas \( (U_i, \varphi_i)_{i \in I} \) obtained from the atlas \( (U_i, \varphi_i)_{i \in I} \) of \( M_n \) by setting \( U_i = U_i \cap \Omega \) and letting \( \varphi_i \) be the restriction of \( \varphi_i \) to \( U_i \).

The sphere \( S_n \) is a compact analytic manifold.

**Proof.** Let us consider the unit sphere \( S_n \subset \mathbb{R}^{n+1} \) centered at \( 0 \in \mathbb{R}^{n+1} \), with \( P \) and \( T \) the north and south poles of coordinates \( z^{n+1} = \pm 1 \), \( z^i = 0 \) for \( 1 \leq i \leq n \) in \( \mathbb{R}^{n+1} \). We define the charts \( (\Omega, \varphi) \) and \( (\Theta, \psi) \) as follows: \( \Omega = S_n \setminus \{P\}, \Theta = S_n \setminus \{T\} \); for \( Q \in \Omega \), \( \varphi(Q) = Q_1 \), the intersection of the straight line \( PQ \) with the hyperplane \( \Pi \) of equation \( z^{n+1} = 0 \) (\( \varphi \) is the stereographic projection of pole \( P \)); and for \( Q \in \Theta \), \( \psi(Q) = Q_2 \), the intersection of the straight line \( TQ \) with \( \Pi \).
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Obviously (0, So) and (8, t1') form an atlas $A$ for $S_2$: $\Omega \cup \Theta = S_2$ and $\Pi$ is identified with $\mathbb{R}^n$. What is the class of $A$?

Let $(r, \alpha)$ be polar coordinates for $Q_1$ and $(\rho, \omega)$ polar coordinates for $Q_2$. Thus $r = OQ_1$, $\rho = OQ_2$, $\alpha = \omega$ and $r\rho = OP.OT = 1$. On $\Omega \cap \Theta$, the function $\varphi \circ \psi^{-1}$, which is defined by $\alpha = \omega$ and $\rho \longrightarrow r = \frac{1}{\rho}$, is analytic since $\rho$ and $r$ are not zero on $\Omega \cap \Theta$. We can see that, in Cartesian coordinates $\{x^i\}$ for $Q_1$, $\{y^i\}$ for $Q_2$ we have $y^i/\rho = x^i/r$; thus

$$y^i = \frac{x^i}{\sum_{j=1}^n (x^j)^2} \quad \text{and} \quad x^i = \frac{y^i}{\sum_{j=1}^n (y^j)^2}.$$

1.9. The real projective space $P_n(\mathbb{R})$ is a compact analytic manifold.

Proof. Recall that $P_n(\mathbb{R}) = (\mathbb{R}^{n+1} - \{O\})/\mathbb{R}$, where $\mathbb{R}$ is the following relation of equivalence: $x \sim \tilde{x}$ in $\mathbb{R}^{n+1} - \{O\}$ if there exists $\rho \in \mathbb{R}$ such that $x = \rho \tilde{x}$.

Let $U_i$ ($i = 1, 2, \cdots, n + 1$) be the set of points of $\mathbb{R}^{n+1}$ whose $i^{th}$ coordinates are not zero.

The open sets $\Omega_i = U_i/\mathbb{R}$ cover $P_n(\mathbb{R})$: $\bigcup_{i=1}^{n+1} \Omega_i = P_n(\mathbb{R})$. We consider the atlas $A = \{\Omega_i, \varphi_i\}_{1 \leq i \leq n+1}$, where $\varphi_i$ is defined as follows.

To the equivalence class $\tilde{x} \in \Omega_i$ ($x = (x^1, x^2, \cdots, x^{n+1})$), we associate $\varphi_i(\tilde{x}) = y_i \in \mathbb{R}^n$ whose coordinates are $y_i^j = x^j/x^i$ for $j < i$ and $y_i^j = x^{j+1}/x^i$ for $j \geq i$, $(y_i^1, y_i^2, \cdots, y_i^{i-1}, 1, y_i^i, \cdots, y_i^n) \in \tilde{x}$. The $\varphi_i$ ($i = 1, 2, \cdots, n + 1$) are bijections; we endow $P_n(\mathbb{R})$ with the topology such that the $\varphi_i$ are homeomorphisms. What is the class of $A$?
To simplify the writing let us consider $\varphi_n \circ \varphi_{n+1}^{-1}$. If $\tilde{x} \in \Omega_n \cap \Omega_{n+1}$, then $x = \{x^i\}$ with $x^n$ and $x^{n+1}$ nonzero. For $j < n$ we have $y_{n+1}^j = x^j/x^{n+1}$ and $y_{n+1}^0 = x^j/x^n$. But $y_{n+1}^n = x^n/x^{n+1} = 1/y_n^0$, so $y_{n+1}^j = y_n^j/y_n^n$ for $j < n$. This change of coordinates is analytic since $y_n^n \neq 0$.

To prove that $P_n(\mathbb{R})$ is compact, we consider in each $\Omega_i$ a compact $K_i$ such that $P_n(\mathbb{R}) = \bigcup_{i=1}^{n+1} K_i$. For instance, let $\tilde{x} \in P_n(\mathbb{R})$, $x = \{x^i\}$ and $x^0$ be such that $|x^i| \leq |x^0|$ for all $1 \leq i \leq n + 1$. In $\Omega_{i_0}$, we consider $K_{i_0}$, the set of $\tilde{x}$ such that $|x^i| \leq |x^0|$ for all $i$. In $\mathbb{R}^n$ the set $K$ defined by $|y^j| \leq 1$ for $1 \leq j \leq n$ is compact; thus $K_{i_0} = \varphi_{i_0}^{-1}(K)$ is compact.

**Partition of Unity**

1.10. To study a manifold, we will have to glue together the charts of an atlas, and use partitions of unity. This is why we will suppose the manifolds *paracompact* (the weakest topological hypothesis which implies the existence of partitions of unity).

A topological space $E$ is *paracompact* if, to any covering of $E$ by open sets $\Omega_i$ ($i \in I$), we can associate a covering, locally finite and thinner, by open sets $\Theta_i$. *Locally finite* means that any point has a neighbourhood $W$ such that $\Omega_i \cap W \neq \emptyset$ except for a finite set of indices $i$. *Thinner* means that $\Theta_i \subseteq \Omega_i$ (with this notation, some $\Theta_i$ may be the empty set). $E$ is *countable at infinity* if there is a family of compact sets $K_q$ ($q \in \mathbb{N}$) such that $K_1 \subset K_2 \subset \cdots \subset K_q \subset K_{q+1} \subset \cdots$, the union of the $K_q$ being $E$ ($E = \bigcup_{q=1}^{\infty} K_q$).

1.11. Theorem. A paracompact manifold is the union of a family of connected manifolds which are countable at infinity.

**Proof.** The manifold is the union of its connected components; let $V_n$ be one of them. Let $\{\Omega_P\}_{P \in V_n}$ be a family of open sets homeomorphic to $\mathbb{R}^n$ with $\overline{\Omega}_P$ compact. Since $V_n$ is paracompact, there is a locally finite cover $\{\Theta_i\}$ thinner than the cover $\{\Omega_P\}$. Pick $\Theta_1 \neq \emptyset$ and set $K_1 = \overline{\Theta}_1$; then by induction we define $K_q = \bigcup_{j \in J_q} \overline{\Theta}_j$, $J_q$ being the set of indices $j$ for which $\Theta_j \cap K_{q-1} \neq \emptyset$. The set $K_q$ is compact, as a finite union of compact sets; indeed we verify below that the set of open sets $\Theta_i$ which have a nonempty intersection with a given compact set $K$ is finite. Let $U_P$ be an open neighbourhood of $P$ such that $\Theta_i \cap U_P = \emptyset$ except for a finite set of indices $i$. The family $\{U_P\}_{P \in V_n}$ covers $K$; thus there exist $U_{P_1}, U_{P_2}, \ldots, U_{P_k}$ which cover $K$. Now if $\Theta_i \cap K \neq \emptyset$, then $\Theta_i \cap U_{P_j} \neq \emptyset$ for at least one $U_{P_j}$, and the set of these $\Theta_i$ is finite.
Moreover \( K_{q-1} \subset \bigcup_{j \in J_q} \Theta_j = \mathcal{K}_q \), so \( W = \bigcup_{q=1}^{\infty} K_q \) is open. But \( W \) is closed too. Indeed, for \( P \in \overline{W} \), let \( \Theta_0 \) be one of the open sets of the covering which is a neighbourhood of \( P \). We have \( W \cap \Theta_0 \neq \emptyset \); thus \( \Theta_0 \) has nonempty intersection with a compact set \( K_q \), and \( P \in \Theta_0 \subset K_{q+1} \subset W = V_n \) since \( V_n \) is connected and \( W \neq \emptyset \). Thus it is proved that \( V_n \) is countable at infinity.

1.12. Theorem (Partition of unity). On a paracompact differentiable manifold of class \( C^k \) (respectively \( C^\infty \)), there exists a \( C^k \) (respectively \( C^\infty \)) partition of unity subordinated to a given covering.

From now on, manifolds are always paracompact; we will mention it no more. Let \( \{\Theta_i\}_{i \in I} \) be a covering of a manifold by open sets. A partition of unity subordinate to the covering \( \{\Theta_i\} \) is a family of functions \( \{\alpha_i\}_{i \in I} \) with the following properties:

i) \( \text{supp} \alpha_i \subset \Theta_i \).

ii) Any point \( P \) has a neighbourhood \( U \) such that \( U \cap \text{supp} \alpha_i = \emptyset \) except for a finite set of \( \alpha_i \).

iii) \( 0 < \alpha_i \leq 1; \sum_{i \in I} \alpha_i = 1 \).

Proof of Theorem 1.12. Let us prove the result for a connected component \( M_n \) of the manifold. According to Theorem 1.11, \( M_n \) is countable at infinity: \( M_n = \bigcup_{p=1}^{\infty} K_p \), \( K_p \subset K_{p+1} \), \( K_p \) being compact. Let \( \{\Theta_i\} \) be the given covering, and for \( Q \in K_{p+1} \backslash K_p \), let us consider a local chart \( (\Theta_f, \varphi_f) \) such that \( \varphi_f^{-1}(B_3) \) is included in an open set \( \Theta_i \); and in \( K_{p+2} \backslash K_{p-1} \) (here we suppose that \( \varphi_f(Q) = 0 \in \mathbb{R}^n \) and \( \varphi_f(\Theta_f) = \mathbb{R}^n \)), \( B_r \subset \mathbb{R}^n \) is the ball centered at 0 of radius \( r \). Now \( \{\varphi_f^{-1}(B_1)\}_{Q \in M} \) is a covering of \( K_1 \); thus a finite subset \( \varphi_f^{-1}(B_1), \ldots, \varphi_f^{-1}(B_1) \) of \( \{\varphi_f^{-1}(B_1)\}_{Q \in M} \) covers \( K_1 \). By induction we exhibit a sequence of points \( Q_k \) such that the open sets \( \varphi_f^{-1}(B_1) \) cover \( K_{p+1} \backslash K_p \); thus \( \varphi_{f_{p+1}}^{-1}(B_1), \ldots, \varphi_{f_{p+1}}^{-1}(B_1) \) cover \( K_{p+1} \backslash K_p \). We verify that the open sets \( \Omega_k = \varphi_{f_k}^{-1}(B_3) \) form a locally finite covering thinner than \( \Theta_i \). Let us consider the function

\[
\gamma(r) = \left[ \int_1^2 p(t)dt \right]^{-1} \int_r^2 p(t)dt
\]

with \( p(t) = e^{1/2} - r^{-1} \) for \( 1 \leq t \leq 2 \) and \( p(t) = 0 \) for \( t \notin [1, 2] \). The function \( \gamma(r) \) is equal to 1 for \( r \leq 1 \) and vanishes for \( r \geq 2 \). Set \( \gamma_f(Q) = \gamma(|\varphi_f(Q)|) \) when \( \varphi_f(Q) \) exists, zero otherwise. The functions \( \alpha_j = \gamma_f/\sum_{i=1}^{\infty} \gamma_i \) form a partition of unity subordinate to the covering \( \{\Theta_f\} \). Since \( \gamma(r) \) is \( C^\infty \), if the manifold is \( C^\infty \) (respectively \( C^k \)), so is the partition of unity; see 1.6.
Differentiable Mappings

1.13. Definition. A $C^k$ mapping $f$ of a differentiable manifold $W_p$ into another $M_n$ is called $C^r$-differentiable ($r \leq k$) at $P \in \Theta \subset W_p$ if $\psi \circ f \circ \varphi^{-1}$ is $C^r$-differentiable at $\varphi(P)$, and we define the rank of $f$ at $P$ to be the rank of $\psi \circ f \circ \varphi^{-1}$ at $\varphi(P)$.

Here $(\Theta, \varphi)$ is a local chart of $W_p$ at $P$, and $(\Omega, \psi)$ is a local chart of $M_n$ with $f(P) \in \Omega$. We easily verify that this definition makes sense, since it does not depend on the local charts.

1.14. Remark. In the definition of a differentiable manifold, when we consider an atlas $(\Theta_i, \varphi_i)_{i \in I}$, each function $\varphi_i$ is a homeomorphism of $\Theta_i$ onto $\varphi_i(\Theta_i) \subset \mathbb{R}^n$. But now $\varphi_i$ is a diffeomorphism of $\Theta_i$ onto $\varphi_i(\Theta_i)$. Indeed, according to Definition 1.13, to see if $\varphi_i$ is a differentiable mapping, we have to verify that $\psi \circ \varphi_i \circ \varphi^{-1}$ is differentiable for a local chart $(\Theta, \varphi)$ and a local chart $(\Omega, \psi)$. Choose $(\Theta, \varphi) = (\Theta_i, \varphi_i)$ and $\psi = \text{Identity}$; then obviously $\varphi_i$ is differentiable: $\text{Id} \circ \varphi_i \circ \varphi_i^{-1} = \text{Id}$, which is differentiable.

1.15. Definition. A $C^r$ differentiable mapping $f$ of $W_p$ into $M_n$ is an immersion if the rank of $f$ is equal to $p = \dim W_p$ at every point $P$ of $W_p$. It is an imbedding if $f$ is an injective immersion such that $f$ is a homeomorphism of $W_p$ onto $f(W_p)$ endowed with the topology induced from $M_n$. A $C^r$ differentiable mapping $f$ of $W_p$ onto $M_n$ is a submersion if the rank of $f$ is equal to $n = \dim M_n$.

Since rank $f \leq \max(p, n)$, it follows that $n \geq p$ if $f$ is an immersion, and $p \geq n$ if $f$ is a submersion.

1.16. Examples. $W_p = \mathbb{R}$, $M_n = \mathbb{R}^2$. 

(i) \hspace{2cm} (ii) \hspace{2cm} (iii)

(iv) \hspace{2cm} (v)
In example (iii) the equation of the mapping for \( t \leq 0 \) is \( x(t) = t, y(t) = 0 \), and for \( t \geq 1 \) it is \( x(t) = 0, y(t) = \frac{1}{t} \). In example (iv) the equation of the mapping is, in polar coordinates, \( \theta = -t, r = \frac{1}{t} \) for \( t > 1 \). (i) is not an immersion, \( \text{rank } f = 0 \) at 0. (ii) is an immersion; we have \( \text{rank } f = 1 \) everywhere; but it is not an injective immersion. (iii) is an injective immersion, but it is not an imbedding. Indeed, \( f((\epsilon, +\epsilon)) \) for any \( \epsilon > 0 \) is not an open set in the induced topology. In the induced topology, a neighbourhood of 0 contains open sets of the form \( f((-\epsilon, +\epsilon)) \cup f((\frac{1}{\epsilon}, \infty)) \).

(iv) is an imbedding. Here there is no difficulty; in the induced topology, a neighbourhood of 0 contains open sets of the form \( f((\frac{1}{\epsilon}, \infty)) \). (v) is an imbedding.

1.17. Definition. Let \( \tilde{M}, M \) be two \( C^k \) differentiable manifolds. \( \tilde{M} \) is called a covering manifold of \( M \) if there exists a differentiable mapping \( \Pi \) (called a projection) of \( \tilde{M} \) onto \( M \), such that for every \( P \in M \):

i) \( \Pi^{-1}(P) \) is a discrete space \( F \), and

ii) there exists a neighbourhood \( \Omega \) of \( P \) such that \( \Pi^{-1}(\Omega) \) is diffeomorphic to \( \Omega \times F \).

Each point \( P' \in \Pi^{-1}(P) \) has a neighbourhood \( \Omega' \subset \tilde{M} \) such that the restriction \( \Pi' \) of \( \Pi \) to \( \Omega' \) is a diffeomorphism of \( \Omega' \) onto \( \Omega \).

The map \( \Pi \) is a 2-sheeted covering if \( F \) consists of two points.
Submanifolds

1.18. Definition. A submanifold of dimension $p$ of a differentiable manifold $M_n$ is a subset $W$ of $M_n$ such that for any point of $W$ there exists a local chart $(\Omega, \varphi)$ of $M_n$, where $\varphi(\Omega)$ is an open set of the form $\Theta \times V$ with $\Theta \subset \mathbb{R}^p$ and $V \subset \mathbb{R}^{n-p}$, such that $\varphi(\Omega \cap W) = \Theta \times \{0\}$. $W$ is endowed with the topology induced from $M_n$.

Thus there exists a system of local coordinates $(x^1, \ldots, x^n)$ on $\Omega$ such that $W_p$ is locally defined by the equations $x^{p+1} = x^{p+2} = \ldots = x^n = 0$. $W_p$ is endowed with a structure of differentiable manifold induced from $M_n$. We consider for $M_n$ an atlas $(\Omega_i, \varphi_i)_{i \in I}$ of local charts as above. Then $(\tilde{\Omega}_i, \tilde{\varphi}_i)_{i \in I}$ is an atlas for $W$ of the same class with $\tilde{\Omega}_i = \Omega_i \cap W$ and $\tilde{\varphi}_i$ the component on $\Theta_i$ of the restriction of $\varphi_i$ to $\tilde{\Omega}_i$.

Considering this definition, it seems difficult to prove that a subset $W$ of $M_n$ is a submanifold. Fortunately there is the following theorem, which is very convenient.

1.19. Theorem. A subset $W$ of $M_n$ defined by a set of $n-p$ equations $f_1(P) = 0, \ldots, f_{n-p}(P) = 0$, where $f_1, \ldots, f_{n-p}$ are $C^1$-functions on $M_n$, is a differentiable submanifold $W_p$ of $M_n$ if the map of $M_n$ into $\mathbb{R}^{n-p}$ defined by $P \mapsto (f_1(P), \ldots, f_{n-p}(P))$ is of rank $n-p$ at any point $P \in W$.

Proof. Let $x_0$ be a point of $W \subset M_n$ and $(\Omega, \varphi)$ a local chart of $M_n$ with $x_0 \in \Omega$, $(x^1, x^2, \ldots, x^n)$ the corresponding coordinates. One of the determinants of the $(n-p) \times (n-p)$ submatrices of the matrix $((\partial f_i/\partial x^j))$ is nonzero at $x_0$. Without loss of generality, let us suppose that it is the one where $j = p+1, p+2, \ldots, n$.

According to the inverse function theorem, there is a neighbourhood $\Theta \subset \Omega$ of $x_0$ on which as coordinates of $P \in \Theta$ we can take $y^1 = x^1, \ldots, y^p = x^p, y^{p+1} = f_1, \ldots, y^n = f_{n-p}$. Let $\psi$ be the homeomorphism defined on $\Theta$ by $P \mapsto \{y^i\} \in \mathbb{R}^n, \psi(\Theta \cap W) \subset \mathbb{R}^p \times \{0\}$. Here $W$ is a differentiable submanifold of $M_n$.

1.20. Example. An open set $\Theta$ of a manifold $M_n$ is a submanifold of dimension $n$. Let us consider an atlas $(\Omega_i, \varphi_i)_{i \in I}$ for $M_n$; then $(\tilde{\Omega}_i, \tilde{\varphi}_i)_{i \in I}$ with $\tilde{\Omega}_i = \Omega_i \cap \Theta$ and $\tilde{\varphi}_i = \varphi_i/\tilde{\Omega}_i$ is an atlas for $\Theta$. $\tilde{\varphi}_i(\tilde{\Omega}_i)$ is an open set of $\mathbb{R}^n$.

The set $S_n(1)$ of the points $x = \{x^i\} \in \mathbb{R}^{n+1}$ satisfying $f_1(x) = \sum_{i=1}^{n+1}(x^i)^2 - 1 = 0$ is a submanifold of $\mathbb{R}^{n+1}$. The rank of the matrix $((\partial f_1/\partial x^j))$ is 1 on $S_n(1)$. Indeed, the derivatives are $\partial f_1/\partial x^j = 2x^j$, and the matrix is never 0 on $S_n(1)$ since $\sum (x^j)^2 = 1$.

The set of matrices $T(n, p)$ with $p$ rows and $n$ columns is a normed vector space. If $a^j_i$ are the components of the matrix $M \in T(n, p)$, we set
\[ \|M\| = \sup |a_i^j|. \]
So the bijection of \( T(n, p) \) onto \( \mathbb{R}^{np} \) defined by \((a_i^j) \rightarrow \{x_k\} \) with \( x_{j+n(i-1)} = a_i^j \) is a homeomorphism. Here one chart covers the manifold \( T(n, p) \).

1.21. The set \( T(n, p, k) \subset T(n, p) \) of matrices of rank \( k \) is a submanifold of \( T(n, p) \).

**Proof.** In order for the rank of \( M \in T(n, p) \) to be greater than or equal to \( k \), we have to have that one of the determinants \( D_\alpha \) of the submatrices \( k \times k \) of \( M \) not zero. The set \( E_\alpha \) of the points satisfying \( D_\alpha \neq 0 \) is an open set of \( T(n, p) \), and their union is an open set of \( T(n, p) \); it is a submanifold of \( T(n, p) \) which is of dimension \( np \) (an open set of a manifold is a submanifold).

Thus if \( k = \inf(n, p) \), we have proved that \( T(n, p, k) \) is a submanifold of \( T(n, p) \) of dimension \( np \).

If \( k < \inf(n, p) \), then at a point \( M \in T(n, p, k) \) at least one of the determinants \( D_\alpha \) is not zero. Let us suppose \( D_1 \neq 0 \); we have to show (and it is a necessary and sufficient condition) that all the \((k+1) \times (k+1)\) determinants of the type

\[
D_\alpha^j(M) = \begin{vmatrix}
a_1^1 & \cdots & a_1^k & a_1^{k+\alpha} \\
\vdots & \ddots & \vdots & \vdots \\
a_k^1 & \cdots & a_k^k & a_k^{k+\alpha} \\
a_k^{k+\beta} & \cdots & a_k^{k+\beta} & a_k^{k+\alpha}
\end{vmatrix}
\]

vanish with \( 1 \leq \alpha \leq n - k \) and \( 1 \leq \beta \leq p - k \).

The map \( \Gamma : T(n, p) \rightarrow \mathbb{R}^{(p-k)(n-k)} \) defined by

\[
\mathcal{M} \rightarrow \{D_{11}(\mathcal{M}), D_{12}(\mathcal{M}), \ldots, D_{n-k,p-k}(\mathcal{M})\}
\]

is of rank \((p-k)(n-k)\) on \( E_1 \). Indeed, the partial derivatives with respect to \( x_\lambda \) with \( \lambda = k + \nu + n(k + \mu - 1) \), \( 1 \leq \nu \leq n - k \), \( 1 \leq \mu \leq p - k \), are

\[
\frac{\partial D_\alpha^j}{\partial x_\lambda} = D_1 \quad \text{if} \ \nu = \alpha \ \text{and} \ \mu = \beta; \quad \frac{\partial D_\alpha^j}{\partial x_\lambda} = 0 \quad \text{otherwise}.
\]

Thus the map \( \Gamma \) is of rank \((n-k)(p-k)\). According to Theorem 1.19, \( E_1 \cap T(n, p, k) \) is a submanifold of dimension \( np - (n-k)(p-k) = k(n+p-k) \). But as \( T(n, p, k) \subset \bigcup E_\alpha \), according to Definition 1.18 (which is purely local) \( T(n, p, k) \) is a submanifold of dimension \( k(n+p-k) \) of \( T(n, p) \).

**The Whitney Theorem**

1.22. **Theorem.** Every paracompact differentiable and connected manifold \( M_n \) may be imbedded in \( \mathbb{R}^{2n+1} \).
We will prove a weaker theorem: every connected $C^2$-differentiable manifold has an immersion in $\mathbb{R}^{2n}$ and an imbedding in $\mathbb{R}^{2n+1}$. The steps of the proof are Propositions 1.25 and 1.27.

In the first lemma we will prove that we can perturb the map $f$ a little bit, locally in $\Omega$, so that the new map is an immersion on $\Omega$. With the second lemma we prove that if $f$ is already an immersion on a compact $K$, we can do this perturbation so that the new map is still an immersion on $K$.

**1.23. Lemma.** Let $f(x)$ be a $C^2$-differentiable map of an open set $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^p$ (with $p \geq 2n$). Then for any $\epsilon > 0$ there exists an $n \times p$ matrix $A = ((a^i_j))$ with $|a^i_j| < \epsilon$ for all $i$ and $j$ so that $x \mapsto f(x) + A.x$ is an immersion.

**Proof.** If $J(x)$ is the Jacobian matrix of $f$ at $x$, we want $J(x) + A$ to be of rank $n$ for all $x \in \Omega$. That is to say, $A$ must be, for any $x$, different from the matrices of the form $B - J(x)$ with $B$ a matrix of rank $< n$.

The map $F$ of $\Omega \times T(n,p,k)$ into $T(n,p)$ defined by $(x,B) \mapsto B - J(x)$ is $C^1$. When $k < n$ the dimension of $\Omega \times T(n,p,k)$ is $n+k(n+p-k) \leq n+(n-1)(p+1) \leq np - 1$ (by setting $k = n - 1$, then taking $p \geq 2n$).

Therefore when $k < n$ the image of $\Omega \times T(n,p,k)$ by $F$ is of zero measure in $T(n,p)$, identified with $\mathbb{R}^{np}$ according to a well known theorem of measure theory:

The image by a $C^1$-mapping of an open set of $\mathbb{R}^n$ into $\mathbb{R}^p$ ($n < p$) is of zero measure in $\mathbb{R}^p$.

Recall that a set $A \subset \mathbb{R}^p$ is said to be of zero measure if for any $\epsilon > 0$, there exists a sequence of balls $B_i$ such that $A \subset \bigcup_{i=1}^{\infty} B_i$ and $\sum_{i=1}^{\infty} \text{vol } B_i < \epsilon$. In particular, if the measure of $A$ is zero, no open set, except the null set, is included in $A$. Thus the interior of $E = F(\Omega \times \bigcup_{k=0}^{p-1} T(n,p,k))$ is empty, and we can choose $A \in T(n,p)$ not in $E$ as close as we want to the zero matrix.

**1.24. Lemma.** Let $f$ be a $C^1$-map of $M_n$ into $\mathbb{R}^p$ ($p > n$). If the rank of $f$ is equal to $n$ on a compact set $K \subset \Omega$, $(\Omega, \varphi)$ a local chart, then there exists $\eta > 0$ such that for any $C^1$-map $g$ satisfying $\|J(g)\| < \eta$ on $K$ we have $f + g$ of rank $n$ on $K$.

**Proof.** We write $J(g)$ for the Jacobian matrix of $g$. Let $\delta(x)$ be the maximum of the absolute values of the determinants of the $n \times n$ submatrices of $J(f)$ at $x$. $\delta(x)$ is positive and continuous on $K$; thus there exists $\delta > 0$ such that $\delta(x) \geq \delta$ for all $x \in K$. A determinant is a continuous function of its components; therefore there is an $\eta > 0$ such that, if $\|A\| < \eta$, the matrix $J(f) + A$ is of rank $n$ on $K$. 


1.25. Proposition. Let $f$ be a $C^k$-map ($2 \leq k \leq \infty$) of the connected and
$C^k$-differentiable manifold $M_n$ into $\mathbb{R}^p$ $(p \geq 2n)$ and let $\psi$ be a continuous
function everywhere positive on $M_n$. There exists a $C^k$-immersion $g$ of $M_n$
into $\mathbb{R}^p$ such that $\|f(P) - g(P)\| \leq \psi(P)$ for all $P \in M_n$.

Proof. Let $(\Omega_j, \varphi_j)$, $j \in \mathbb{N}$, be a sequence of local charts such that the $\Omega_j$
homeomorphic to $B_3$, form a locally finite cover of $M_n$, like $V_j = \varphi_j^{-1}(B_1)$.
For the notations, we refer to the proof of Theorem 1.12. Let us apply
Lemma 1.23 to $f \circ \varphi_1^{-1}$ on $B_3$ with a small enough $\varepsilon$. We set $f_1(P) = f(P) + \gamma_1(P)A_1 \circ \varphi_1(P)$. Since $\gamma_1 \leq 1, \gamma_1 = 1$ on $V_1$ and $\gamma_1 = 0$ outside $\varphi_1^{-1}(B_2)$,
it follows that $f_1$ is of rank $n$ on $\overline{V}_1$ and $\|f_1(P) - f(P)\| \leq \psi(P)/2$, because
we have chosen $\varepsilon$ small enough for that.

By induction we define a sequence of $C^k$-differentiable maps $f_r$ of rank
$n$ on $K_r = \bigcup_{j=1}^r \overline{V}_j$. We set
$$f_{r+1}(P) = f_r(P) + \gamma_{r+1}(P)A_{r+1} \circ \varphi_{r+1}(P),$$
the $A_{r+1}$ being chosen according to Lemma 1.23 with $\Omega = \Omega_{r+1}$ and $\varepsilon$ small
enough so that $\|f_{r+1}(P) - f_r(P)\| \leq 2^{-r-1}\psi(P)$ and so that $f_{r+1}$ is still of
rank $n$ on $K_r \cap \overline{\Omega}_{r+1}$ (Lemma 1.24).

The covering being locally finite, at a given point $P$, from some $j_0$ on we
have $f_j(P) = f_{j+1}(P)$ for $j \geq j_0$. Thus the limit of the sequence $\{f_j\}$ is a $C^k$-
differentiable map $g$ which is an immersion satisfying $\|f(P) - g(P)\| \leq \psi(P)$
for all $P \in M_n$.

1.26. Theorem. Every connected, paracompact, $C^k$-differentiable mani-
fold has a $C^k$-immersion in $\mathbb{R}^{2n}$. The image of the manifold may be in
a ball $B_\rho \subset \mathbb{R}^{2n}$ of radius $\rho > 0$ as small as one wants.

Proof. We consider the map $f$ of $M_n$ into $\mathbb{R}^{2n}$ defined by $f(P) = 0$ for all
$P \in M_n$, and we choose $\psi(P) = \rho$. According to Proposition 1.25, there
exists an immersion $g$ of $M_n$ into $\mathbb{R}^{2n}$ such that $g(M_n) \subset B_\rho$.

1.27. Proposition. Let $f$ be a $C^k$-differentiable map ($2 \leq k \leq \infty$) of the
connected $C^k$-differentiable manifold $M_n$ into $\mathbb{R}^p$ $(p \geq 2n+1)$ and let $\psi$ be a continuous
everywhere positive function on $M_n$. Then there is an injective
$C^k$-immersion $h$ of $M_n$ into $\mathbb{R}^p$ satisfying $\|f(P) - h(P)\| \leq \psi(P)$ for all
$P \in M_n$.

Proof. According to Proposition 1.25, there is an immersion $g$ of $M_n$ into
$\mathbb{R}^{2n}$ such that $\|f(P) - g(P)\| \leq \psi(P)/2$ for all $P \in M_n$. According to The-
orem 1.19, $g$ is locally injective: every point $P \in M_n$ has a neighbourhood
$\Omega_P$ where $g$ is injective.
Let \( \{ \Omega_i \}, \ i \in \mathbb{N} \), be a sequence of such compact neighbourhoods which form a locally finite covering of \( M_n \) (as in Proposition 1.25), \( \Omega_i \) being homeomorphic to \( B_3 \). Let \( \{ b_i \} \) be a sequence of points of \( \mathbb{R}^{2n+1} \) which we will choose later. Let us say already that we will choose \( b_i \) satisfying

\[
\|b_i\| \leq 2^{-(i+2)} \inf_{P \in \Omega_i} \psi(P)
\]

and small enough so that the maps \( h_r = g + \sum_{i=1}^{r-1} b_i \gamma_i \) are immersions according to Lemma 1.24.

Let \( D_r \subset M \times M \) be the set of pairs \((P_1, P_2)\) for which \( \gamma_r(P_1) \neq \gamma_r(P_2) \).

We consider the map \( G_r \) of the open set \( D_r \) into \( \mathbb{R}^{2n+1} \) defined by

\[
G_r : (P_1, P_2) \rightarrow \frac{h_r(P_1) - h_r(P_2)}{\gamma_r(P_2) - \gamma_r(P_1)}.
\]

\( G_r \) is \( C^2 \)-differentiable and the dimension of \( D_r \) is \( 2n \); therefore \( G_r(D_r) \) is of zero measure in \( \mathbb{R}^{2n+1} \). So we can successively choose \( b_r \) not in \( G_r(D_r) \) and close enough to \( 0 \in \mathbb{R}^{2n+1} \), as we said above. The covering \( \{ \Omega_i \} \) being locally finite, at every point \( P \), from some \( i_0 \) on, we have \( h_i(P) = h_{i+1}(P) \) for \( i \geq i_0 \).

The limit of the sequence \( h_i \) is a \( C^k \)-differentiable immersion \( h \) which is injective. Indeed, by contradiction, let us suppose that there exist \( P_1 \) and \( P_2 \) such that \( h(P_1) = h(P_2) \).

Let \( j_0 \) be an integer such that for \( j \geq j_0 \) we have \( \gamma_j(P_1) = \gamma_j(P_2) \) and \( h_j(P_1) = h(P_1) = h(P_2) = h_j(P_2) \). Now \( h_{r+1}(P_1) = h_{r+1}(P_2) \) gives \( h_r(P_1) + b_r \gamma_r(P_1) = h_r(P_2) + b_r \gamma_r(P_2) \). Since \( b_r \notin G_r(D_r) \), this implies that \( \gamma_r(P_1) = \gamma_r(P_2) \) and \( h_r(P_1) = h_r(P_2) \). By induction, for any \( i \), we have \( h_i(P_1) = h_i(P_2) \) and \( \gamma_i(P_1) = \gamma_i(P_2) \).

At least one \( \gamma_i(P_1) \) is not zero, say \( \gamma_{i_0}(P_1) \neq 0 \). Thus \( \gamma_{i_0}(P_2) \) is not zero. So \( P_1 \) and \( P_2 \) belong to \( \Omega_{i_0} \), where \( g = h_1 \) is injective, which is in contradiction with \( h_1(P_1) \neq h_1(P_2) \) if \( P_1 \neq P_2 \).

1.28. Theorem. Every \( C^k \)-differentiable manifold \( (2 \leq k \leq \infty) \) has a \( C^k \)-imbedding into \( \mathbb{R}^{2n+1} \), and its image is a closed subset of \( \mathbb{R}^{2n+1} \).

Proof. According to Proposition 1.27, there is an injective immersion \( h \) of \( M_n \) into \( \mathbb{R}^{2n+1} \). In order that \( h \) be an imbedding, it is necessary and sufficient that for any compact set \( K \subset \mathbb{R}^{2n+1} \), \( h^{-1}(K) \) is compact. Indeed, since \( h \) is injective, \( h \) will be then a homeomorphism of \( M_n \) onto \( h(M_n) \).

We will choose \( f \) and \( \psi \) for that. Pick \( \psi \equiv 1 \) and all the components of \( f(P) \) zero except the first; we choose \( f_1(P) = \sum_{j=1}^{\infty} j \gamma_j(P) \). We have that \( h(P) \in K \) implies \( \|h(P)\| \leq \rho = \sup \|x\| \) for \( x \in K \) and \( \|f(P)\| \leq \rho + 1 \). Thus \( P \in \bigcup_{1 \leq j \leq \rho+1} V_j \), which is compact (\( \gamma_j \) being equal to 1 on \( V_j \), \( P \notin V_j \) when \( j > \rho + 1 \)).
The Sard Theorem

1.29. Definition. Let $M_n$ and $W_p$ be two $C^\infty$ differentiable manifolds of dimension $n$ and $p$ respectively, and let $f$ be a map of class $C^k$ of $M_n$ into $W_p$. The points of $M_n$ where $\text{rank } f < p$ are called critical points of $f$. All other points of $M_n$ are called regular points. A point $Q \in W_p$ such that $f^{-1}(Q)$ contains at least one critical point is called a critical value. All other points of $W_p$ are called regular values.

When $f$ is a smooth real valued function ($W_p = \mathbb{R}$), $P$ is a critical point of $f$ if all the first derivatives of $f$ at $P$ vanish, since $(\text{rank } f)_P = 0$. The gradient at $P$ vanishes. A critical point $P$ is called non-degenerate if and only if the matrix $((\partial^2 f / \partial x^i \partial x^j))_P$ is non-singular.

The index of $f$ at $P$ is the number of negative eigenvalues of $((\partial^2 f / \partial x^i \partial x^j))_P$. M. Morse proved that any bounded smooth function $f : M \rightarrow \mathbb{R}$ can be uniformly approximated by smooth functions which have no degenerate critical point (see Milnor [8]).

If $n < p$, all points of $M_n$ are critical since $\text{rank } f \leq \inf(n,p) < p$, and the critical values form a set of measure zero. This is obvious according to Proposition 0.28.

But the Sard Theorem asserts that this result holds in general if the map $f$ is $C^\infty$: the set of critical values is of measure zero. We are at the beginning of the course (in Chapter 1) and we have only seen some definitions and a few theorems, but nevertheless we were able to prove a very important theorem, the Whitney Theorem, which shows that a differentiable manifold, whose definition is abstract, is nothing else than a surface of dimension $n$ in $\mathbb{R}^p$ for $p$ large enough. And now it is possible to give a second very important theorem:

1.30. Theorem (The Sard theorem). Let $M_n$ and $W_p$ be two connected $C^\infty$ differentiable manifolds of dimension $n$ and $p$ respectively, $n \geq p$, and let $f$ be a map of class $C^k$ ($k \geq 1$) of $M_n$ into $W_p$. If $k \geq n - p + 1$, the critical values form a set of measure zero.

In case the manifolds are only $C^r$, we have Sard's theorem for any $p$ if and only if $r \geq n$.

Proof. Since the manifolds are separable (they are connected), we have only to prove the result for a local chart $(\Omega, \varphi)$ of $M$ and a local chart $(\Theta, \psi)$ of $W$ such that $f(\Omega) \subset \Theta$.

Thus the theorem will be proved if the result holds when $M$ is an open set of $\mathbb{R}^n$ and $W = \mathbb{R}^p$. Indeed, we will only have to consider the map $\psi \circ f \circ \varphi^{-1}$ of $\phi(\Omega) \subset \mathbb{R}^n$ into $\psi(\Theta) \subset \mathbb{R}^p$. 

Thus henceforth \( f \) is a \( C^k \) map with \( k \geq \max(n - p + 1, 1) \) of an open set \( \Omega \subset \mathbb{R}^n \) into \( \mathbb{R}^p \), and \( A \) is the set of the critical points of \( f \). When \( n < p \) we proved the result in 0.28. Therefore we assume \( n \geq p \).

**First step.** The Sard theorem is true for functions \( (p = 1) \).

As in the proof of Proposition 0.28, we only have to prove the result when \( M_n \) is a unit closed cube \( C \) of \( \mathbb{R}^n \). The proof is by induction on the dimension \( n \). Suppose \( n = 1 \) and \( f \in C^1 \). Then \( x \in \Omega \subset \mathbb{R}^n \) is a critical point of \( f \) if and only if \( f'(x) = 0 \). Moreover, as \( f \) is \( C^1 \) on the compact set \( C \), for any \( \epsilon > 0 \) there exists \( m \in \mathbb{N} \) such that \( |y - z| < 1/m \) implies \( |f'(y) - f'(z)| < \epsilon \).

Thus \( |y - x| < 1/m \) implies \( |f'(y)| < \epsilon \), and according to the mean value theorem \( |f(x) - f(y)| < \epsilon/m \).

Now we proceed as in the proof of Proposition 0.28. We divide \( C \) into \( m \) intervals of length \( 1/m \). Let \( J \) be one of them which has a critical point. We have \( \text{meas } f(J) \leq \epsilon/m \). The set \( A \) of critical points of \( f \) is covered by some intervals like \( J \). The number of these intervals is at most \( m \), of course. So \( \text{meas } f(A) \leq m\epsilon/m = \epsilon \). Since \( \epsilon \) is as small as one wants, \( \text{meas } f(A) = 0 \).

The result is true for \( n = 1 \). Now we suppose, by induction, that if \( \Theta \) is an open set in \( \mathbb{R}^{n-1} \), and \( g \) is a \( C^{n-1} \) map of \( \Theta \) into \( \mathbb{R} \), then \( g(B) \) has measure zero, \( B \) being the set of critical points of \( g \). The result is also true if \( \Theta \) is a \( C^\infty \) separable manifold, according to the beginning of the proof.

Let \( A_k \) \( (1 \leq k \leq n) \) be the set of critical points \( x \) of \( f \) such that all derivatives of \( f \) of order less than or equal to \( k \) vanish at \( x \), with \( x \notin A_{k+1} \) when \( k < n \). If \( x \in A_n \), for any \( \epsilon > 0 \) there exists \( m \in \mathbb{N} \) such that \( |y - x| \leq n^{1/2}/m \) implies that the norm of the differential \( D^nf(y) \) of order \( n \) at \( y \) is less than \( \epsilon \). Thus \( |y - x| \leq n^{1/2}/m \) implies

\[
||f(y) - f(x)|| \leq \epsilon m^{-n} n^{n/2}.
\]

Then we split the unit cube \( C \) into \( m^n \) cubes of side \( 1/m \). If \( K \), one of them, has a point \( x \in A_n \), we have

\[
\text{meas } f(K) \leq 2\epsilon m^{-n} n^{n/2}.
\]

At most \( m^n \) little cubes have a non-empty intersection with \( A_n \); thus \( \text{meas } f(A_n) \leq 2\epsilon n^{n/2} \). Since \( \epsilon \) is as small as one wants, \( \text{meas } f(A_n) = 0 \).

When \( x \in A_k \) \( (k < n) \), at least one derivative of \( f \) of order \( k + 1 \) does not vanish at \( x \). Suppose without loss of generality that \( \partial_1 h(x) \neq 0 \) with \( h \) some derivative of \( f \) of order \( k \), \( h(x) = 0 \). The function \( h \), which is at least \( C^1 \), has rank 1 at \( x \), so in a neighbourhood \( \Theta \) of \( x \) the set \( D = \{ y \in \Theta \mid h(y) = 0 \} \) is a submanifold of dimension \( n - 1 \). Obviously \( A_k \cap \Theta \subset D \). Let \( g = f/D \); by the induction hypothesis \( \text{meas } g(A_k \cap \Theta) = 0 \). The result follows since \( \mathbb{R}^n \) is separable.
Second step. Let
\[ A_j = \{ x \in \Omega \mid \text{rank } Df(x) = j \}, \quad A = \bigcup_{0 \leq j < p} A_j. \]

Since \( \mathbb{R}^n \) is separable, we have only to show that for any \( j \) any \( x_o \in A_j \) has a neighbourhood \( \Theta \subset \mathbb{R}^n \) such that \( \text{meas } f(\Theta \cap A_j) = 0 \). Observe that \( \bigcup_{0 \leq i \leq j} A_i \) is closed for each \( j \). Hence \( A_0 \) is measurable; thus \( A_1 \) is measurable and by induction any \( A_j \) is measurable. By the following lemma, \( f(\Theta \cap A_o) \) has measure zero.

1.31. Lemma. Let \( f : \Omega \to \mathbb{R}^p \) be a \( C^k \) map (\( k \geq n \)) and let \( A_o = \{ x \in \Omega \mid Df(x) = 0 \} \). Then \( f(A_o) \) has measure zero in \( \mathbb{R}^p \).

Proof. Let \( f_1(x) \) be the first coordinate of \( f(x) \). We have \( A_o \subset B = \{ x \in \Omega \mid df_1(x) = 0 \} \); hence \( f(A_o) \subset f_1(B) \times \mathbb{R}^{p-1} \). By the first step, \( f_1(B) \) has measure zero in \( \mathbb{R}^n \), so that \( f_1(B \times \mathbb{R}^{p-1}) \) has measure zero in \( \mathbb{R}^p \) according to Fubini’s theorem.

1.32. Remark. When the regularity of \( f \) is only \( k = \max(n - p + 1, 1) \), we can prove the result of Lemma 1.31 by using Lemmas 1.33 and 1.34, but the proof is not so easy.

Third part of the proof of Sard’s theorem. Let \( 0 < j < p \) and \( x_o \in A_j \) be such that \( \text{rank } Df(x_o) = j \). We can suppose without loss of generality that \( ((\partial f_{i\alpha}/\partial x^i))_{x_o} \) does not vanish. According to the inverse function Theorem 0.29, in a neighbourhood of \( x \), \( \{ y^i \} \) with \( y^i = f_i \) for \( 1 \leq i \leq j \) and \( y^i = x^i \) for \( j < i \leq n \) is a coordinate system. Set \( F(y) = f(x) \), where \( y = \varphi(x) \), \( \varphi \) being the diffeomorphism of change of coordinate systems. We have \( F(y) = f_i(x) = y^i \) for \( 1 \leq i \leq j \). \( F(y) \) is a \( C^k \) map of a neighbourhood \( V \) of \( y_o = \varphi(x_o) \) into \( \mathbb{R}^n \). Since the rank of \( F \) at \( y_o \) is \( j \), \( DF(y_o) \) is represented by a matrix \( ((a_{il})) \) such that \( a_{il} = (\partial F_{i\alpha}/\partial y^i)(y_o) = \delta_i^\alpha \) for \( 1 \leq \alpha \leq j \). Since the rank of this matrix is \( j \), we have \( a_{ul} = 0 \) for \( i > j \) and \( l > j \). Hence if we consider the map
\[ G : \mathbb{R}^{p-j} \ni V \ni a \to (F_{j+1}(a, z), F_{j+2}(a, z), \ldots, F_p(a, z)), \]
where \( a \in \mathbb{R}^j \) and \( V_a = \{ z \in \mathbb{R}^{n-j} \mid (a, z) \in V \} \), then \( DG(z_o) = 0 \) if \( (a, z_o) = y_o \). Let \( E_j \) be the set of critical points of \( F \), where the rank of \( F \) is \( j \). Up to the diffeomorphism of change of coordinate systems, \( E_j \) is nothing else than \( A_j \cap \varphi^{-1}(V) \). By Lemma 1.31, \( F(V_a \cap E_j) \) has measure zero in \( \mathbb{R}^{p-j} \). Hence \( \text{meas } F(E_j) = 0 \) according to Fubini’s theorem. This implies
\[ \text{meas } f(A_j \cap \varphi^{-1}(V)) = 0, \]
and then \( \text{meas } f(A_j) = 0 \). The proof of Sard’s theorem is complete when \( k \geq n \). If not, we have two more lemmas to prove.
1.33. Lemma. Let $f$ be a function of class $C^k$ on a closed unit cube in $\mathbb{R}^n$, and $A$ the set of critical points of $f$. Then $A = A_0 \cup \tilde{A}$ with $A_0$ countable and $\tilde{A}$ such that, for any pair $(x, y)$ in $\tilde{A}$,

$$|f(x) - f(y)| \leq a(\|x - y\|)\|x - y\|^k,$$

where $a(\epsilon) \to 0$ as $\epsilon \to 0$.

Proof. If $n = 1$, $A$ is the set of points $x$ where $f'(x) = 0$. Let $A_0^1 \subset A$ be the set of isolated points ($A_0^1$ is countable). If $x \in A - A_0^1$, there is a sequence of points $x_i$ in $A$ such that $x_i \to x$. Also, $f'(x_i) = 0$ implies $f''(x) = 0$. Let $A_0^2$ be the set of isolated points $x \in A - A_0^1$ where $f''(x) = 0$ ($A_0^2$ is countable) and so on until we consider $f^k(x)$. Then $A_0 = \bigcup_{1 \leq i \leq k} A_0^i$ is countable, $A_0^1$ being the set of isolated points $x$ in $A - \bigcup_{1 \leq i \leq k} A_0^i$ where $f^i(x) = 0$. We set $\tilde{A} = A - A_0$. At $x \in \tilde{A}$ we have $f^j(x) = 0$ ($1 \leq j \leq k$). Using the mean value theorem, since $y \to f^k(y)$ is continuous, we get the inequality of the lemma. We can extend the proof in the case $n > 1$ (see Sternberg [14]).

1.34. Lemma. Let $f$ be a $C^k$ map of a closed unit cube in $\mathbb{R}^n$ into $\mathbb{R}^p$ and let $A$ be the set of points where $f$ has rank zero. Then $f(A)$ has measure zero if $k \geq n/p$.

Proof. We saw that $A = A_0 \cup \tilde{A}$ (Lemma 1.33). $f(A_0)$ has measure zero since $A_0$ is countable. For $\tilde{A}$ we split the unit cube into $m^n$ cubes $K$ of side $1/m$ ($m \in \mathbb{N}$). If $x, y$ in $\tilde{A}$ belong to the same little cube $C$, then $\|y - x\| \leq n^{1/2}/m = \epsilon$. Hence $\|f(x) - f(y)\| \leq a(\epsilon)\epsilon^k$, and

$$\text{meas } f(C \cap \tilde{A}) \leq \omega_{p-1}a^p(\epsilon)\epsilon^{kp}.$$ 

Since there are at most $m^n$ cubes $K$ with $K \cap \tilde{A} \neq \emptyset$,

$$\text{meas } f(\tilde{A}) \leq \omega_{p-1}a^p(\epsilon)\epsilon^{kp-nn/2}.$$ 

When $kp \geq n$, the right hand side is as small as one wants. Hence $\text{meas } f(\tilde{A}) = 0$.

1.35. Remarks. When $k \geq \max(n - p + 1, 1)$, we have $k \geq n/p$, since $n - p + 1 \geq n/p$ if $n \geq p$. The assumption on $k$ in Sard's theorem is sharp; Whitney (see Sternberg [14]) gave a counterexample in the case when $k < \max(n - p + 1, 1)$.

1.36. Corollary. The set of regular values of a $C^\infty$ map $f$ of $M_n$ onto $W_p$ is everywhere dense in $W_p$. If $Q$ is a regular value, then $f^{-1}(Q)$ is a submanifold of $M_n$ of dimension $n - p$.

$V = f^{-1}(Q)$ is defined locally by a set of $p$ functions $(f_1, f_2, \ldots, f_p) = f$, and we know that rank $f = p$; so we have to consider a local chart at $Q$. 
Thus, according to Theorem 1.19, $V$ is a submanifold of $M_n$ of dimension $n-p$.

Exercises and Problems

1.37. Exercise. $C = \{z \in \mathbb{C}||z|| = 1\}$ being the unit circle centered at 0 in $\mathbb{C}$, we consider the map $\varphi$ of $\mathbb{R}$ into the torus $T = C \times C$ defined by $\mathbb{R} \ni u \rightarrow \varphi(u) = (e^{2i\pi u}, e^{2i\pi u}) \in T$. What can we say about $\varphi$ (immersion, injective immersion, imbedding) and about the subset $\varphi(\mathbb{R})$ of $T$?

1.38. Exercise. On $\mathbb{R}^2$, endowed with the coordinate system $(x, y)$, we consider the function $f$ defined by

$$f(x, y) = x^3 + xy + y^3 + 1.$$  

a) For which points $P \in \mathbb{R}^2$ is $f^{-1}[f(P)]$ a submanifold imbedded in $\mathbb{R}^2$?

b) Draw the complementary set of this set of points $P$.

1.39. Exercise. Let $M, W$ be two differentiable manifolds. We suppose $W$ is compact and $M$ is connected. We consider a differentiable map $\Pi$ of $W$ into $M$ such that $\Pi$ is locally a diffeomorphism at any point $Q \in W$.

a) Let $P \in M$, and show that $\Pi^{-1}(P)$ is a finite subset of $W$.

b) Prove that the cardinality of $\Pi^{-1}(P)$ does not depend on $P$.

c) Verify that $W$ is a covering manifold of $M$.

d) Construct a counterexample which will establish that if $W$ is not compact, $W$ may be not a covering manifold of $M$. One will choose $M = \mathbb{R}$.

1.40. Exercise. We consider the map $\varphi$ of $\mathbb{R}^{n+1}$, endowed with the coordinate system $\{x^i\}$ $(i = 1, 2, \ldots, n+1)$, into $\mathbb{R}^{2n+1}$, whose coordinates are $\{y^\alpha\}$ $(\alpha = 1, 2, \ldots, 2n+1)$, defined by

$$\varphi: \{x^i\} \rightarrow \{y^\alpha\} \quad \text{with} \quad y^\alpha = \sum_{k+l=1+\alpha} x^k x^l$$

for $1 \leq \alpha < 2n+1$, and $y^{2n+1} = \sum_{i=1}^{n+1} (x^i)^2$.

a) On which open set of $\mathbb{R}^{n+1}$ is $\varphi$ an immersion?

b) From $\varphi$, construct an imbedding of $\mathbb{P}_n(\mathbb{R})$ into $\mathbb{R}^{2n}$.

1.41. Exercise. On $\mathbb{C}^{m+1} \backslash \{0\}$ consider the equivalence relation $\mathcal{R}$ defined by $z_1 \sim z_2$ if there exists $\rho \in \mathbb{C}$, $\rho \neq 0$, such that $z_1 = \rho z_2$. The quotient set of $\mathbb{C}^{m+1} \backslash \{0\}$ by $\mathcal{R}$ is called the complex projective space $\mathbb{P}^m(\mathbb{C})$. As we did for the real projective space, we define an atlas $(\Omega_i, \varphi_i)$ ($i = 1, 2, \ldots, m+1$) as follows. Let $U_i$ be the set of points of $\mathbb{C}^{m+1} \backslash \{0\}$ whose $i^{th}$ complex coordinate is not zero, and let $\Omega_i = U_i/\mathcal{R}$. Let $\tilde{z}$ be the equivalence class of
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a point \( z \in U_i \), and let \((\xi_1^1, \xi_2^1, \ldots, \xi_{i-1}^1, 1, \xi_{i+1}^1, \ldots, \xi_{m+1}^1) \in \tilde{z} \). \( \varphi_i(\tilde{z}) \) will be the point \( \xi \) of \( \mathbb{C}^m \) of complex coordinates \((\xi_1^1, \xi_2^1, \ldots, \xi_{i-1}^1, \xi_{i+1}^1, \ldots, \xi_{m+1}^1) \).

a) We define a subset \( M \) of \( \mathbb{P}^3(\mathbb{C}) \) by

\[
M = \{ \tilde{z} \in \mathbb{P}^3(\mathbb{C}) \mid \xi_2^2 \xi_3^3 = \xi_1^1 \xi_4^4, (\xi_2^2)^2 = \xi_1^1 \xi_3^3, (\xi_3^3)^2 = \xi_2^2 \xi_4^4 \}
\]

where \( z = (\xi_1^1, \xi_2^2, \xi_3^3, \xi_4^4) \in \tilde{z} \). Show that \( M \) is a compact submanifold of dimension 2 of \( \mathbb{P}^3(\mathbb{C}) \).

b) Consider the map \( \Phi : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^3(\mathbb{C}) \) defined by

\[
\mathbb{P}^1(\mathbb{C}) \ni \tilde{z} \rightarrow \Phi(\tilde{z}) = \tilde{z} \in \mathbb{P}^3(\mathbb{C})
\]

with \( z = (\eta_1^1, \eta_2^2) \in \mathbb{C}^2 \setminus \{0\} \) and

\[
z = [(\eta_1^1)^3, (\eta_1^1)^2 \eta_2^2, \eta_1^1 (\eta_2^2)^2, (\eta_2^2)^3] \in \mathbb{C}^4 \setminus \{0\}.
\]

Prove that \( \Phi \) is a diffeomorphism of \( \mathbb{P}^1(\mathbb{C}) \) onto \( M \).

1.42. Exercise. Let \( M_n \) be a \( \mathcal{C}^\infty \) compact manifold of dimension \( n \).

a) Exhibit a finite cover of \( M_n \) by a family of open sets \( \Omega_i \) \((i = 1, 2, \ldots, p)\) homeomorphic to a ball of \( \mathbb{R}^n \) such that \( \overline{\Omega_i} \subset \Omega_i \) with each \( \Omega_i \) homeomorphic to a ball of \( \mathbb{R}^n \). Prove the existence of \( \mathcal{C}^\infty \) functions \( f_i \) on \( M_n \) satisfying \( 0 \leq f_i \leq 1 \), \( \text{supp} \, f_i \subset \Omega_i \) and \( f_i(x) = 1 \) when \( x \in \Omega_i \).

b) Deduce the existence of an imbedding \( \psi \) of \( M_n \) into \( \mathbb{R}^q \) with \( q = (n + 1)p \).

1.43. Exercise. Show that a proper injective immersion is an imbedding.

1.44. Exercise. We identify \( \mathbb{R}^4 \) with the set of \( 2 \times 2 \) matrices.

a) Show that the set \( M_2 \) of \( 2 \times 2 \) matrices whose determinant is equal to 1 is a submanifold of \( \mathbb{R}^4 \). What is its dimension?

b) Prove that the tangent space to \( M_2 \) at \( I_2 = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \) may be identified with the set of matrices of zero trace.

c) Show that the set \( M_n \) of \( n \times n \) matrices whose determinant is equal to 1 may be identified with a submanifold of \( \mathbb{R}^{n^2} \). What is its dimension?

d) Characterize the tangent space to \( M_n \) at the unit \( n \times n \) matrix \( I_n \).

1.45. Exercise. Let \( E \) be the set of straight lines in \( \mathbb{R}^3 \).

a) Establish a bijection between \( E \) and the quotient set of \( \mathbb{P}_2 \times \mathbb{R}^3 \) by an equivalence relation (\( \mathbb{P}_2 \) the real projective space of dimension 2).

b) We endow \( E \) with the structure of a topological space (the finest possible) such that \( \pi \) is continuous. Is \( \pi \) open? (A map is open if the image of any open set is an open set.)

c) Show that \( E \) is Hausdorff.
d) On $E$ define a structure of an analytic manifold. 

*Hint.* Consider the open sets $\Theta_i = \pi(\Omega_i \times \mathbb{R}^3)$, where the open sets $\Omega_i$ ($i = 1, 2, 3$) cover $\mathbb{P}_2$ as in the course.

e) Using the proof of the previous question, show that $E$ is a vector fiber bundle. Characterize its elements. What is its dimension?

f) Let $S$ be the unit sphere in $\mathbb{R}^3$. Prove that the set of tangent straight lines to $S$ is a compact differential submanifold of $E$. What is its dimension?

*Hint.* Apply in $\theta_i$ a theorem of the course. For the compactness look at the proof of compactness of $\mathbb{P}_2$.

1.46. Exercise. Let $M_p$ and $W_n$ be two $C^\infty$ differentiable manifolds of dimension $p$ and $n$ respectively, and let $f$ be a $C^\infty$ map from $M_p$ into $W_n$. We say that $f$ is a *subimmersion* at $x \in M_p$ if there exist a neighbourhood $\Omega \subset M_p$ of $x$, a neighbourhood $\Theta \subset W_n$ of $f(x)$, a $C^\infty$ manifold $V$, and two $C^\infty$ maps $g$ and $h$ such that $g$ is a submersion of $\Omega$ into $V$ and $h$ is an immersion of $V$ into $W_n$ with $f/\Omega = h \circ g$.

a) Prove that $f$ is a subimmersion if and only if the rank of $f$ is constant in a neighbourhood of $x$.

b) Show that the function $M_p \ni x \rightarrow r(x) = \text{rank of } f \text{ at } x$ is lower semicontinuous.

c) Prove that the subset of $M_p$ where $f$ is a subimmersion is dense in $M_p$.

Solutions to Exercises

Solution to Exercise 1.37.

$\varphi$ is everywhere of rank 1; thus $\varphi$ is an immersion. If $\varphi(u) = \varphi(v)$, we would have $u - v \in Z$ in order that $e^{2i\pi u} = e^{2i\pi v}$, but also $u - v \in \pi Z$ in order that $e^{2iu} = e^{2iv}$. Since $\pi$ is irrational, this is impossible except if $u = v$.

Therefore $\varphi$ is an injective immersion. Let $P = (e^{2iu}, e^{2iv})$ be a point of $\varphi(\mathbb{R})$. If $\varphi$ is an imbedding, there would exist an open set $\Omega$ of $T$ whose intersection with $\varphi(\mathbb{R})$ would be a given connected arc $\gamma$ of $\varphi(\mathbb{R})$ through $P$. We can show that, as close as one wants to $P$, there are points of $\varphi(\mathbb{R})$ which do not belong to $\gamma$. It is well known that for any $\alpha > 0$, there are integers $n$ and $p$ such that $|p\pi - n| < \alpha$. Set $u = u + n$, $e^{2i\pi u} = e^{2i\pi v}$ and $|e^{2iv} - e^{2iu}| = |e^{2in} - 1| = |e^{2i(n - p\pi)} - 1| \leq |e^{2i\alpha} - 1|$, which is as small as one wants.

Finally, let us prove the well known fact mentioned above. For $p \in \mathbb{N}$ we set $x_p = \inf(p\pi - n)$ for $n \in \mathbb{N}$, $n < p\pi$. So $x_p = p\pi - n_p$, and the set $\{x_p\}_{p \in \mathbb{N}} \subset [0, 1]$. 

Let \( \{x_{p_i}\} \) be a subsequence which converges in \([0, 1]\). So we have 
\[(p_{i+1} - p_i) \pi - (n_{p_{i+1}} - n_{p_i}) \to 0.\]

**Solution to Exercise 1.38.**

If \((x, y) \to f(x, y)\) is everywhere of rank 1 on \(W_p = f^{-1}(f(P))\), then \(W_p\) is a submanifold imbedded in \(\mathbb{R}^2\) according to Theorem 1.19. Now \(\partial f/\partial x = 3x^2 + y\) vanishes when \(y = -3x^2\), and \(\partial f/\partial y = 3y^2 + x\) is zero when \(x = -3y^2\). These two equations have for solution \((x = 0, y = 0)\) and 
\((x = -\frac{1}{3}, y = -\frac{1}{3})\). Set \(\Omega = (0, 0)\) and \(Q = (-\frac{1}{3}, -\frac{1}{3})\). Then \(f(0, 0) = 1\) and 
\(f(-\frac{1}{3}, -\frac{1}{3}) = 1 + 1/27\).

Let us study \(W_\Omega\). \(W_\Omega \setminus \Omega\) is a submanifold. We have to see what happens at \(\Omega\). The equation of \(W_\Omega\) is \(x^3 + xy + y^3 = 0\). If \(x\) is small with respect to \(y\), the equation is \(xy + y^3 = 0\). \(y = 0\) is impossible if \(x \neq 0\), but we have an arc where \(x \sim -y^2\). By symmetry we have an arc where \(y \sim -x^2\). \(\Omega\) is a double point. At \(\Omega\) there are two arcs, one with tangent \(y = 0\), the other with tangent \(x = 0\). There is nothing more in the neighbourhood of \(\Omega\), since if we suppose \(x \sim ay\) with \(a \neq 0\), we find \(ay^2 = 0\), which is impossible. Let us study \(W_Q\). \(W_Q \setminus Q\) is a submanifold. In a similar way as above, we prove that \(Q\) is an isolated point. So \(W_Q\) is not a submanifold.

Therefore the set of points \(P\) for which \(W_P\) is a submanifold is \(A = f^{-1}((-\infty, 1) \cup (1, 1 + 1/27) \cup (1 + 1/27, \infty))\).

Let \(D\) be the line of equation \(x + y = 1/3\). \(f(D) = 1 + 1/27\). The complementary set of \(A\) is \(W_\Omega \cup W_Q\). That is, \(\{Q\} \cup D\) and a curve through \(\Omega\) asymptotic to \(D\). This set is symmetric with respect to the line \(y = x\).
Solution to Exercise 1.39.

$\Pi^{-1}(P)$ is compact (a closed subset of a compact set), and $\Pi^{-1}(P)$ is a set of isolated points since $\Pi$ is locally a diffeomorphism. Thus $\Pi^{-1}(P)$ is a finite subset of $W$. We will show below that $\text{card} \; \Pi^{-1}(P)$ is locally constant. This will imply that $\text{card} \; \Pi^{-1}(P) = Constant$, since $V$ is connected. Let $P_i$ ($i = 1, 2, \ldots, m$) be the points of $\Pi^{-1}(P)$. There are disjoint neighbourhoods $\Omega_i$ of $P_i$ which are homeomorphic to a neighbourhood $\Theta$ of $P$. Let $\{Q_k\}$ be a sequence in $\Theta$ which converges to $P$. Obviously $\limsup_{k \to \infty} \text{card} \; \Pi^{-1}(Q_k) \geq \text{card} \; \Pi^{-1}(P)$. Let us prove the equality by contradiction. If we do not have equality, there is a sequence $\{x_k\}$ in $W$ such that $\Pi(x_k) = Q_k$ and $x_k \not\in \Omega_i$ for all $k$ and $i$.

Since $W$ is compact, a subsequence, noted always $\{x_k\}$, converges to a point $x$ of $W$. $x$ is not a point $P_i$, and, by the continuity of $\Pi$, $\Pi(x) = P$, a contradiction. $\Pi^{-1}(\Theta)$ is diffeomorphic to $F \times \Theta$, $F$ a set of $m$ points. For a counterexample take $W = ]-\infty, 0[ \cup ]1, \infty[$ and $\Pi$ the identity map.

Solution to Exercise 1.40.

We have $\partial y^{n+1}/\partial x^i = 2x^i$ and $\partial y^\alpha/\partial x^i = 2x^{1+\alpha-i}$ when $\alpha - n \leq i \leq \alpha$. We verify that $\varphi$ is of rank $n + 1$ on $\mathbb{R}^{n+1} \setminus \{0\}$. If $x^1 \neq 0$, the first $(n + 1) \times (n + 1)$ determinant in $D\varphi$ is equal to $(2x^1)^{n+1} \neq 0$. If $x^1 = 0$ and $x^2 \neq 0$, the first row is zero but the $(n + 1) \times (n + 1)$ determinant in $D\varphi$ with the $n + 1$ following rows is equal to $(2x^2)^{n+1} \neq 0$, and so on.

Let $S_n$ be the unit sphere in $\mathbb{R}^{n+1}$ and let $\tilde{\varphi}$ be the restriction of $\varphi$ to $S_n$. We consider the map $f$ of $S_n$ into $\mathbb{R}^{2n}$ defined by $f^\alpha(P) = \tilde{\varphi}^\alpha$ for $1 \leq \alpha \leq 2n$.

On $S_n$, $y^{n+1} = 1$. Thus $\text{rank} \; f = \text{rank} \; \tilde{\varphi} = n$. Let $\Pi$ be the natural projection $S_n \to \mathbb{P}_n(\mathbb{R})$. We verify easily that if $f(P) = f(Q)$, then $Q = P$ or $Q$ is the point opposite to $P$ on $S_n$. Thus we can write $f = \tilde{f} \circ \Pi$, with $\tilde{f}$ an injective immersion of $\mathbb{P}(\mathbb{R})$ into $\mathbb{R}^{2n}$. Since $\tilde{f}$ is obviously proper ($\mathbb{P}_n(\mathbb{R})$ is compact), $\tilde{f}$ is a homeomorphism of $\mathbb{P}_n(\mathbb{R})$ on its image.
Chapter 2

Tangent Space

In this chapter we introduce many basic notions. First we will study tangent vectors, then differential forms. We will give two different definitions of a tangent vector at a point $P \in M_n$ (they are dual to each other). Then, of course, we will prove that the definitions are equivalent. $M_n$ is a $C^r$ differentiable manifold ($r \geq 1$) and $(\Omega, \varphi)$ a local chart at $P$; $\{x^i\}$ are the corresponding coordinates.

We define the tangent space $T_p(M), P \in M$. It is the set of the tangent vectors $X$ at $P$, which has a natural vector space structure of dimension $n$. The union of all tangent spaces is the tangent bundle $T(M)$. We will show that if $r > 1$, $T(M)$ carries a structure of differentiable manifold of class $C^{r-1}$, which is a vector fiber bundle $(T(M), \pi, M)$ of fiber $\mathbb{R}^n$ and basis $M$ $(T(M) \ni X \to \pi(X) = P \in M$ if $X \in T_p(M))$. Likewise we define the cotangent bundle $T^*(M)$. A vector field on $M$ is a differentiable map $\xi$ of $M$ into $T(M)$ such that $\pi \circ \xi$ is the identity. Thus a vector field $X$ on $M$ is a mapping that assigns to each point $P \in M$ a vector $X(P)$ of $T_p(M)$, an assignment which satisfies some regularity condition.

Likewise we define differential $p$-forms, exterior differential $p$-forms, . . .

The notions of linear tangent mapping $(\Phi_*)_P$ and linear cotangent mapping $(\Phi^*)_P$ associated to a differentiable map $\Phi$ of one differentiable manifold into another are very important.

The linear cotangent mapping $\Phi^*$ allows us to transport differentiable $p$-forms in the direction opposite to that of the map $\Phi$.

This chapter continues with the definition of the bracket $[X, Y]$ of two vector fields $X$ and $Y$. 
2. Tangent Space

We will define the exterior product, the inner product and the exterior differential on the direct sum of exterior differential forms; our definition is an extension to exterior differential forms of the usual differential of differentiable functions. We proceed with the study of orientable manifolds and of manifolds with boundary, and conclude with Stokes' formula.

Tangent Vector

2.1. Definition. Consider differentiable maps \( \gamma_i \) of a neighbourhood of \( 0 \in \mathbb{R} \) into \( M_n \) such that \( \gamma(0) = P \). Let \( (\Omega, \varphi) \) be a local chart at \( P \).

We say that \( \gamma_1 \sim \gamma_2 \) if \( \varphi \circ \gamma_1 \) and \( \varphi \circ \gamma_2 \) have the same differential at zero. We verify that this definition makes sense (it does not depend on the local chart). It is an equivalence relation \( \mathcal{R} \). A tangent vector \( X \) at \( P \) to \( M_n \) is an equivalence class for \( \mathcal{R} \).

2.2. Definition. Let us consider a differentiable real-valued function \( f \) defined on a neighbourhood \( \theta \) of \( P \in \Omega \). We say that \( f \) is flat at \( P \) if \( d(f \circ \phi^{-1}) \) is zero at \( \phi(P) \).

This definition makes sense; it does not depend on the local chart. If \( (\tilde{\Omega}, \tilde{\varphi}) \) is another local chart at \( P \), then, on \( \Omega \cap \tilde{\Omega} \),

\[
d(f \circ \tilde{\varphi}^{-1}) = d(f \circ \varphi^{-1}) \circ d(\varphi \circ \tilde{\varphi}^{-1}).
\]

A tangent vector at \( P \in M_n \) is a map \( X : f \rightarrow X(f) \in \mathbb{R} \) defined on the set of the differentiable functions in a neighbourhood of \( P \), where \( X \) satisfies the following conditions:

a) If \( \lambda, \mu \in \mathbb{R} \), then \( X(\lambda f + \mu g) = \lambda X(f) + \mu X(g) \).

b) \( X(f) = 0 \) if \( f \) is flat at \( P \).

It follows from a) and b) that
c) \( X(fg) = f(P)X(g) + g(P)X(f) \).

Indeed,

\[
X(fg) = X\left\{ [f - f(P) + f(P)][g - g(P) + g(P)] \right\}, \\
X(fg) = X\left\{ (f - f(P))(g - g(P)) \right\} + f(P)X(g) + g(P)X(f)
\]

since \( X(1) = 0 \) (the constant function 1 is flat).

Now \( d((f_1g_1) \circ \varphi^{-1})_{\varphi(P)} = d((f_1 \circ \varphi^{-1})(g_1 \circ \varphi^{-1}))_{\varphi(P)} = 0 \) if \( f_1 \) and \( g_1 \) are zero at \( P \). Thus \( (f - f(P))(g - g(P)) \) is flat at \( P \), and c) follows.

2.3. Definition. The tangent space \( T_P(M) \) at \( P \in M_n \) is the set of tangent vectors at \( P \).

Using Definition 2.2, let us show that the tangent space of Definition 2.3 has a natural vector space structure of dimension \( n \). We set

\[
(X + Y)(f) = X(f) + Y(f) \quad \text{and} \quad (\lambda X)(f) = \lambda X(f).
\]

With this sum and this product, \( T_P(M) \) is a vector space. And now let us exhibit a basis. \( \{x^i\} \) being the coordinate system corresponding to \((\Omega, \varphi)\), we define the vector \( (\partial/\partial x^i)_P \) by

\[
\left( \frac{\partial}{\partial x^i} \right)_P (f) = \left[ \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} \right]_{\varphi(P)}.
\]

The vectors \( (\partial/\partial x^i)_P \) \((1 \leq i \leq n)\) are independent since \( (\partial/\partial x^i)_P (x^j) = \delta^j_i \), and they form a basis. Indeed, as \( f - \sum_{i=1}^n (\partial f/\partial x^i)_P x^i \) is flat at \( P \),

\[
X(f) = \left[ \sum_{i=1}^n X(x^i) \left( \frac{\partial}{\partial x^i} \right)_P \right] (f).
\]

The \( X^i = X(x^i) \) are the components of \( X \) in the basis \( (\partial/\partial x^i)_P \). Observe that the expression of \( X(f) \) contains only the first derivatives of \( f \).

2.4. Proposition. The two definitions of a tangent vector are equivalent.
Let $\gamma(t)$ be a map in the equivalence class $\bar{\gamma}$ ($\gamma(0) = P$), and $f$ a real-valued function in a neighbourhood of $P$.

Considering the map $X : f \mapsto [\partial(f \circ \gamma)/\partial t]_{t=0}$, we define a map $\Psi$ of the set of tangent vectors (Definition 2.1) to the set of tangent vectors (Definition 2.2), $\Psi : \bar{\gamma} \mapsto X$. Indeed, since

$$d(f \circ \gamma) = d(f \circ \varphi^{-1} \circ \varphi \circ \gamma) = d(f \circ \varphi^{-1}) \circ d(\varphi \circ \gamma),$$

if $\gamma_1 \sim \gamma_2$ we have

$$\left( \frac{\partial(f \circ \gamma)}{\partial t} \right)_{t=0} = \left( \frac{\partial(f \circ \gamma_2)}{\partial t} \right)_{t=0},$$

because by definition $[d(\varphi \circ \gamma_1)]_{t=0} = [d(\varphi \circ \gamma_2)]_{t=0}$. Moreover, $X$ is a tangent vector (Definition 2.2): (a) is obvious, and if $f$ is flat at $P$, then $\left( \partial(f \circ \gamma)/\partial t \right)_{t=0} = 0$ since $(d(f \circ \varphi^{-1})\varphi(P) = 0$.

Let us show now that $\Psi : \bar{\gamma} \mapsto X$ is one-to-one and onto. Let $X$ be a tangent vector (Definition 2.2), $X = \sum_{i=1}^{n} X^i(\partial/\partial x^i)_P$. Consider the map $\gamma : (-\epsilon, \epsilon) \ni t \mapsto \gamma(t) \in M_n$, the point whose coordinates are $\{tX^i\}$ (we suppose that $\varphi(P) = 0 \in \mathbb{R}^n$). Then

$$\left( \frac{\partial(f \circ \gamma)}{\partial t} \right)_{t=0} = \sum_{i=1}^{n} \frac{\partial(f \circ \varphi^{-1})}{\partial x^i} \frac{\partial(tX^i)}{\partial t} = X(f).$$

So $\Psi$ is onto. Moreover, if $\gamma_1$ is not equivalent to $\gamma_2$, then $[d(\varphi \circ \gamma_1)]_{t=0} \neq [d(\varphi \circ \gamma_2)]_{t=0}$, and it is possible to exhibit a function $f$ such that

$$[d(f \circ \gamma_1)]_{t=0} \neq [d(f \circ \gamma_2)]_{t=0}.$$

2.5. Definition. The tangent bundle $T(M)$ is $\bigcup_{P \in M} T_P(M)$. If $T^*_P(M)$ denotes the dual space of $T_P(M)$, the cotangent bundle $T^*(M)$ is $\bigcup_{P \in M} T^*_P(M)$. If $r > 1$, we will show that $T(M)$ carries a structure of differentiable manifold of class $C^{r-1}$. Likewise for $T^*_P(M)$.

Linear Tangent Mapping

2.6. Definition. Let $\Phi$ be a differentiable map of $M_n$ into $W_p$ (two differentiable manifolds). Let $P \in M_n$, and set $Q = \Phi(P)$. The map $\Phi$ induces a linear map $(\Phi_*)_P$ of the tangent bundle $T_P(M)$ into $T_Q(W)$ defined by

$$[(\Phi_*)_P X](f) = X(f \circ \Phi);$$

here $X \in T_P(M)$, $(\Phi_*)_P X \in T_Q(W)$ and $f$ is a differentiable function in a neighbourhood $\theta$ of $Q$. We call $(\Phi_*)_P$ the linear tangent mapping of $\Phi$ at $P$.

To define a vector of $T_Q(W)$ (Definition 2.2), we must specify how it acts on differentiable functions defined in a neighbourhood of $Q$. Obviously, $f \mapsto [(\Phi_*)_P X](f)$ is linear. Moreover, if $f$ is flat at $Q$, then $f \circ \Phi$, which is
differentiable in a neighbourhood of \( P \), is flat at \( P \), and we have \( X(f \circ \Phi) = 0 \). So \((\Phi_*)_P\) is a linear map of \( T_P(M) \) into \( T_Q(W) \).

\[
(\Phi_*)_P \text{ is nothing else than } (d\Phi)_P. \text{ Indeed, consider a local chart at } P \text{ with coordinates } \{x^i\} \text{ and a local chart at } Q \text{ with coordinates } \{y^\alpha\}. \Phi \text{ is defined in a neighbourhood of } P \text{ by } p \text{ real-valued functions } \Phi^\alpha(x^1, x^2, \ldots, x^n), \alpha = 1, 2, \ldots, p. \text{ Using intrinsic notations to simplify, we get }
\]

\[
X(f \circ \Phi) = d(f \circ \Phi)_P \circ X = (df) \circ (d\Phi)_P \circ X = (df) \circ (\Phi_*)_P X.
\]

Indeed, \( \{X^i\} \) being the components of \( X \) in the basis \( \{(\partial/\partial x^i)_P\} \), the components of \( Y = (\Phi_*)_P X \) are

\[
Y^\alpha = \sum_{i=1}^{n} \frac{\partial \Phi^\alpha}{\partial x^i} X^i
\]
in the basis \( \{(\partial/\partial y^\alpha)_Q\} \). When we use intrinsic notation, we do not specify the local charts. In the coordinate systems \( \{x^i\} \) and \( \{y^\alpha\} \), the equality above shows that \( (d\Phi)_P = ((\partial \Phi^\alpha/\partial x^i))_P = (\Phi_*)_P \). When we do not specify the point \( P \), we write \( \Phi_* \) instead of \((\Phi_*)_P\).

2.7. Definition. Linear cotangent mapping \((\Phi^*)_P\). Let \( P \in M_n \) and \( Q = \Phi(P) \). By duality, we define the linear cotangent mapping \((\Phi^*)_P\) of \( T_Q^*(W) \) into \( T_P^*(M) \) as follows:

\[
T_Q^*(W) \ni \omega \longrightarrow (\Phi^*)_P(\omega) \in T_P^*(M),
\]

\[
\langle (\Phi^*)_P(\omega), X \rangle = \langle \omega, (\Phi_*)_P(X) \rangle \text{ for all } X \in T_P(M).
\]

In case \( \omega = df \) we saw (Definition 2.6) that \( d(f \circ \Phi)_P \circ X = (df) \circ (\Phi_*)_P X \).

Thus

\[
(\Phi^*)_P(df) = d(f \circ \Phi)_P.
\]

2.8. Proposition. \( \Psi_* \circ \Phi_* = (\Psi \circ \Phi)_* \).

Let \( V \) be a third differentiable manifold and \( \Psi \) a differentiable mapping of \( W \) into \( V \). If \( f \) is a differentiable function in a neighbourhood of \( \Psi(Q) \)
and $X \in T_P(M)$, then
\[
[\Psi_* (\Phi_* X)] (f) = [\Phi_* (X)] (f \circ \Psi) = X (f \circ \Psi \circ \Phi) = [(\Psi \circ \Phi)_* (X)] (f).
\]
If $\Phi$ is a diffeomorphism, we infer that $\Phi_*$ is bijective and $(\Phi^{-1})_* = (\Phi_*)^{-1}$.

2.9. Example. The tangent vector $\frac{d \gamma}{dt}$ to a differentiable curve $\gamma(t)$ of $M_n$ ($\gamma$ is a differentiable map of $(a, b) \subset \mathbb{R}$ into $M_n$). Let $t_0 \in (a, b)$. By definition $(\frac{d \gamma}{dt})_{t_0}$ is the tangent vector at $\gamma(t_0)$ defined by $(\frac{d \gamma}{dt})_{t_0} = [\gamma_* (\frac{d}{dt})]_{t_0} = (\frac{d}{dt})$ being the unit vector on $\mathbb{R}$. For a differentiable function $f$ in a neighbourhood of $\gamma(t_0)$, we have
\[
\left( \frac{d \gamma}{dt} \right)_{t_0} (f) = \lim_{h \to 0} \frac{f[\gamma(t_0 + h)] - f[\gamma(t_0)]}{h}.
\]

Vector Bundles

2.10. Proposition. The tangent bundle $T(M)$ has a structure of differentiable manifold of class $C^{r-1}$, if $M_n$ is a differentiable manifold of class $C^r$ with $r > 1$.

Let $(U, \varphi)$ be a local chart on $M_n$ and $P \in U$. If $\{x^i\}$ are the coordinates of $Q \in U$ and $\{\xi^i\}$ the coordinates of $\varphi(Q) \in \mathbb{R}^n$, then $x^i = \xi^i$ for $1 \leq i \leq n$. This is the equality of two real numbers. If we consider the equality of two functions, we must write $x^i = \xi^i \circ \varphi$. We have
\[
\varphi_* \left( \frac{\partial}{\partial x^i} \right)_P = \left( \frac{\partial}{\partial \xi^i} \right)_{\varphi(P)}.
\]

Indeed, $f$ being a differentiable function in a neighbourhood of $\varphi(P)$ in $\mathbb{R}^n$,
\[
\left[ \varphi_* \left( \frac{\partial}{\partial x^i} \right)_P \right] (f) = \left( \frac{\partial}{\partial x^i} \right)_P (f \circ \varphi) = \left( \frac{\partial f}{\partial \xi^i} \right)_{\varphi(P)} = \left( \frac{\partial}{\partial \xi^i} \right)_{\varphi(P)} (f).
\]
Thus $\varphi_*$ is a bijection of $T(U)$ onto $\varphi(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$.

Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ be an atlas for $M_n$. The set of $[T(U_\alpha), (\varphi_\alpha)_*]_{\alpha \in A}$ is an atlas for $T(V)$. Let us show that this atlas is of class $C^{r-1}$. Suppose $U_\alpha \cap U_\beta \neq \emptyset$. On $\varphi_\alpha(U_\alpha \cap U_\beta)$ set $\theta = \varphi_\beta \circ \varphi_\alpha^{-1}$; then we have $\theta_\ast(Q, X) = [\theta(Q), (d\theta)_Q(X)]$, where $Q \in \varphi_\alpha(U_\alpha \cap U_\beta)$ and $X \in T_Q(\mathbb{R}^n)$. Thus $(d\theta)_Q$ is of class $C^{r-1}$.

2.11. Definition. A differentiable manifold $E$ is a vector fiber bundle of fiber the vector space $F$ if there exist a differentiable manifold $M$ (called the basis) and a differentiable map $\Pi$ of $E$ on $M$ such that, for all $P \in M$, $\Pi^{-1}(P) = E_P$ is isomorphic to $F$ and there exist a neighbourhood $U$ of $P$ in $M$ and a diffeomorphism $\rho$ of $U \times F$ onto $\Pi^{-1}(U)$ whose restriction to each $E_P$ is linear, $\rho$ satisfying $\Pi \circ \rho(P, z) = P$ for all $z \in F$. 

2.12. Proposition. The tangent bundle $T(M)$ is a vector bundle of fiber $\mathbb{R}^n$.

$M$ is the basis. If $X \in T(M)$ and $X \in T_P(M)$ for a unique point $P \in M$, then the map $\Pi$ is $X \mapsto \Pi(X) = P$. Thus $\Pi^{-1}(P) = T_P(M)$, which is a vector space of dimension $n$: $F = \mathbb{R}^n$. If $(U, \varphi)$ is a local chart at $P$, we saw that $\varphi_*$ is a diffeomorphism of $T(U) = \Pi^{-1}(U)$ onto $\varphi(U) \times \mathbb{R}^n$. So we can choose $\rho = \varphi_*^{-1} \circ (\varphi, \text{Id})$, and we know that $(\varphi_*^{-1})_P$ is linear. Moreover, $\rho(P, z) \in T_P(M)$, and thus $\Pi \circ \rho(P, z) = P$, for all $z \in \mathbb{R}^n$.

2.13. Definition. Likewise we can consider the fiber bundles $T^*(M)$, $\wedge^p T^*(M)$, $T^*_r(M)$:
$T^*(M) = \bigcup_{P \in M} T^*_P(M)$,
$\bigwedge^p T^*(M) = \bigcup_{P \in M} \bigwedge^p T^*_P(M)$,
where $\bigwedge^p T^*_P(M)$ is the space of skew-symmetric $p$-forms on $T_P(M)$, and
$T^*_r(T(M)) = \bigcup_{P \in M} \bigotimes^r T^*_P(M) \otimes T_P(M)$,
where $\bigotimes^r T^*_P(M) \otimes T_P(M)$ is the space of tensors of type $(r, s)$, $r$ times covariant, $s$ times contravariant, on $T_P(M)$.

2.14. Definition. A section of a vector fiber bundle $(E, \Pi, M)$ is a differentiable map $\xi$ of $M$ into $E$ such that $\Pi \circ \xi = \text{identity}$.

A vector field is a section of $T(M)$.

An $(r, s)$-tensor field is a section of $T^*_r(T(M))$.

An exterior differential $p$-form is a section of $\bigwedge^p T^*(M)$. In a local chart an exterior differential $p$-form

$$\eta = \sum_{1 \leq j_1 < j_2 < \cdots < j_p \leq n} a_{j_1 \cdots j_p} dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_p},$$

where $a_{j_1 \cdots j_p}$ are real valued functions.

$\bigwedge^p(M)$ will denote the vector space of exterior differential $p$-forms.

$\Gamma(M)$ will denote the vector space of vector fields.

$T^*_r(M)$ will denote the vector space of $(r, s)$-tensor fields.

The Bracket $[X, Y]$

2.15. Definition. The bracket $[X, Y]$ of two vector fields $X$ and $Y$ is the vector field defined by

$$[X, Y](f) = X[Y(f)] - Y[X(f)],$$
where \( f \) is a \( C^2 \) function on \( M \). Let us show that, in fact, the definition is valid for \( C^1 \) functions and that we have defined above a new vector field \([X, Y]\). Using intrinsic notations,

\[
[X, Y](f) = d[Y(f)].X - d[X(f)].Y = d[df].X - df.d[X(f)].Y
\]

\[
= (d^2 f)(X, Y) + df.dY.X - (d^2 f)(Y, X) - df.dX.Y
\]

\[
= df[dY.X - dX.Y] = [dY.X - dX.Y](f),
\]

since \( d^2 f \) is a symmetric bilinear form. From this expression, it is clear that \([X, Y]\) satisfies conditions (a) and (b) of Definition 2.2. If the vector fields \( X \) and \( Y \) are \( C^r \), then \([X, Y]\) is a \( C^{r-1} \) vector field. In a coordinate system \( \{x^i\} \) corresponding to a local chart \((\Omega, \varphi)\),

\[
[X, Y] = \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} \left( X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \right] \frac{\partial}{\partial x^i}.
\]

2.16. Definition. **Lie algebra.** The map \((X, Y) \rightarrow [X, Y]\) is bilinear and antisymmetric.

\(\alpha) [X, Y] = -[Y, X], \text{ and } [X, Y] \text{ satisfies the Jacobi identity.}\)

\(\beta) [X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.\)

A real vector space \( L \), endowed with a bilinear map \( L \times L \) into \( L \) satisfying \(\alpha) \) and \(\beta) \), is called a **Lie algebra**. So the set of \( C^\infty \) vector fields is a Lie algebra. A straightforward computation proves \(\beta) \), and \(\alpha) \) is obvious.

2.17. Definition. **Projectable vector field.** Let \( M \) and \( W \) be two differentiable manifolds and \( \Psi \) a differentiable map of \( M \) into \( W \). We have defined the linear tangent mapping, but in most cases it does not allow us to associate to a vector field \( X \) on \( M \) a vector field on \( W \). There are two reasons for this. First, \( \Psi(M) \) may be not all \( W \); second, if \( \Psi \) is not injective, we may have at some points of \( W \) several tangent vectors images by \( \Psi_* \) of vectors of the vector field \( X \).

This is why we say that a vector field \( X \) is **projectable** by \( \Psi \), if for all \( Q \in \Psi(M) \) and each \( P_i \in M \) such that \( \Psi(P_i) = Q \), we have \( (\Psi_*)_P X(P_i) \) independent of \( i \). \( \Psi(M) \) is assumed to be \( W \).

If \( \Psi \) is a diffeomorphism, any vector field \( X \) on \( M \) is projectable by \( \Psi \).

Let \( Y \) be a vector field on \( W \). The vector fields \( X \) and \( Y \) are said to be **compatible** by \( \Psi \) if for all \( P \in M \) we have \((\Psi_*)_P X(P) = Y(\Psi(P))\).

2.18. Proposition. **Let** \( X_1, Y_1 \) and \( X_2, Y_2 \) **be two pairs of vectors fields compatible by** \( \Psi \). **Then** \([X_1, X_2] \) **and** \([Y_1, Y_2] \) **are compatible by** \( \Psi \) **(see 2.17 for the notations).** **Consider** \( P \in M \) **and** \( Q = \Psi(P), \) **and let** \( g \) **be a** \( C^2 \)
function in a neighbourhood of $Q$. Then

$$[Y_1, Y_2](g) = Y_1[Y_2(g)] - Y_2[Y_1(g)] = X_1[X_2(g \circ \Psi)] - X_2[X_1(g \circ \Psi)]$$

$$= [X_1, X_2](g \circ \Psi).$$

Thus $\Psi_*([X_1, X_2]) = [\Psi_*(X_1), \Psi_*(X_2)].$

In the proof above there are some subtle points. When we write for the definition of the linear tangent mapping $Y_2(g) = X_2(g \circ \Psi)$, we understand the equality of two real numbers. But here they are functions. So we must write

$$[Y_2(g)] \circ \Psi = X_2(g \circ \Psi).$$

2.19. Definition. A differentiable manifold $M_n$ of dimension $n$ is parallelisable if on $V$ there are $n$ vector fields $X_1, X_2, \ldots, X_n$, such that, at any point $P \in M_n$, $\{X_1(P), X_2(P), \ldots, X_n(P)\}$ is a basis for $T_P(M)$.

2.20. Definition. The product manifold $M \times W$ of two differentiable manifolds $M$ and $W_p$ is a differentiable manifold of dimension $n + p$, defined by the atlas $(U_i \times \Omega_j, (\varphi_i, \psi_j)) (i, j) \in I \times J)$ on the topological product space $M \times W$, $(U_i, \varphi_i)_{i \in I}$ being an atlas on $M_n$ and $(\Omega_j, \psi_j)_{j \in J}$ an atlas on $W_p$.

2.21. Proposition. A manifold $M_n$ is parallelisable if and only if its tangent bundle $T(M)$ is trivial—that is to say, diffeomorphic to $M \times \mathbb{R}^n$, the diffeomorphism $\rho$ being linear on each fiber and satisfying $\Pi \circ \rho^{-1}(P, z) = P$ (see 2.12).

Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{R}^n$ and $\Phi$ the diffeomorphism of $T(M)$ onto $M \times \mathbb{R}^n$ if $T(M)$ is trivial.

Set $X_i(P) = \Phi^{-1}(P, e_i)$ for all $P \in M_n$ and $i = 1, 2, \ldots, n$. Then the set $\{X_i(P)\}$, $i = 1, 2, \ldots, n$, is a basis of $T_P(M)$, as otherwise a linear combination $\overline{e} \neq 0$ of $\overline{e_i}$ would be such that $\Phi^{-1}(P, \overline{e}) = 0$. That is impossible, since $\Phi^{-1}$ is a diffeomorphism and $\Phi^{-1}(P, 0) = 0$. So $M_n$ is parallelisable.

Conversely, if there exist $n$ vector fields $X_i$ ($i = 1, 2, \ldots, n$) such that for all $P \in M_n$ the set $\{X_i(P)\}$ is a basis of $T_P(M)$, let us consider the map $\theta$ of $M \times \mathbb{R}^n$ into $T(M)$ defined by

$$\theta\left(P, \sum_{i=1}^{n} \lambda^i \overline{e_i}\right) = \sum_{i=1}^{n} \lambda^i X_i(P),$$

where the $\lambda^i$ are $n$ real numbers.

$\theta$ is a differentiable bijection, linear for fixed $P$. If we prove that the rank of $\theta$ is $2n$, then $\theta$ is a diffeomorphism, $\Phi = \theta^{-1}$ and $T(M)$ is trivial. Let
{x^i} be a coordinate system in a neighbourhood of P. \{x^i\} and \{\lambda^j\} form a coordinate system in a neighbourhood of (P, \mathbb{R}^n) in M \times \mathbb{R}^n. Moreover,

\[ X_i(x) = \sum_{k=1}^{n} X_i^k(x) \frac{\partial}{\partial x_k} \]

and

\[ (D\theta)_p = \begin{pmatrix} (\text{Id}) & (0) \\ (A) & (X_i^k(P)) \end{pmatrix}. \]

Here (Id) is the n \times n identity matrix, (0) is the n \times n zero matrix, (A) is an n \times n matrix, and the n \times n matrix (X_i^k(P)), with components X_i^k(P), is invertible. Thus the rank of \((D\theta)_p\) is 2n.

**Exterior Differential**

2.22. Definition. The algebra of exterior differential forms \(\Lambda(M)\).

\(\Lambda(M) = \bigoplus_{p=0}^{n} \Lambda^p(M)\), with an exterior product defined below, is an algebra. For simplicity we will say only "differential forms" instead of "exterior differential form" when no confusion is possible. We suppose \(M\) is a \(C^\infty\) manifold. The algebras \(\Lambda^0(M) = \mathcal{C}^\infty(M)\) and \(\Lambda^p(M)\) for \(p > 0\) are defined in 2.14. Given \(\eta \in \Lambda^q(M)\) and \(\xi \in \Lambda^p(M)\), we define \(\eta \wedge \xi \in \Lambda^{p+q}(M)\), the exterior product of \(\eta\) and \(\xi\), by

\[
(\eta \wedge \xi)(X_1, \ldots, X_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in \mathcal{P}} \epsilon(\sigma) \eta(X_{\sigma(1)}, \ldots, X_{\sigma(q)}) \xi(X_{(1+q)}, \ldots, X_{(p+q)}),
\]

where \(X_1, \ldots, X_{p+q}\) are \(p + q\) vector fields and the sum is over the set \(\mathcal{P}\) of permutations \(\sigma\), \(\epsilon(\sigma)\) being the signature of \(\sigma\). The exterior product is associative and anticommutative: \(\xi \wedge \eta = (-1)^{pq} \eta \wedge \xi\).

We also define the inner product \(i(X)\eta\) of a differential form \(\eta \in \Lambda^q(M)\) (1 \(\leq q \leq n\) ) by a vector field \(X\). \(i(X)\eta\) is a differential \((q-1)\)-form defined as follows: If \(X_i (i = 1, 2, \ldots, q-1)\) are \(q-1\) vector fields, then

\[
[i(X)\eta](X_1, X_2, \ldots, X_{q-1}) = \eta(X, X_1, X_2, \ldots, X_{q-1}).
\]

We verify that if \(\xi \in \Lambda^p(M)\), then

\[
i(X)(\eta \wedge \xi) = [i(X)\eta] \wedge \xi + (-1)^q \eta \wedge [i(X)\xi] \quad \text{and} \quad i(X)[i(X)\eta] = 0.
\]

2.23. Proposition. Let \(\Phi\) be a differentiable map of \(M_n\) into \(W_p\). To any differential \(q\)-form \(\eta\) on \(W\), we can associate a differential \(q\)-form \(\Phi^*\eta \in \Lambda^q(M)\), the inverse image of \(\eta\) by \(\Phi\), defined by

\[
(\Phi^*\eta)_p(X_1, X_2, \ldots, X_q) = \eta_{\Phi(p)}(\Phi_* X_1, \ldots, \Phi_* X_q),
\]
$X_1, \ldots, X_q$ being $q$ vectors of $T_p(M)$. For a function $f$ on $W$ ($f \in \Lambda^0(W)$, we set $\Phi^*f = f \circ \Phi$.

Here there is no difficulty such as in Definition 2.17. Indeed if $P \in M$, then $Q = \Phi(P)$ is unique and $(\Phi^*\eta)_P = (\Phi_P^*)\eta_Q$. We verify that

$$\Phi^*(\eta \wedge \xi) = \Phi^*\eta \wedge \Phi^*\xi.$$ 


To $\eta \in \Lambda^q(M)$, we associate the exterior differential form $d\eta \in \Lambda^{q+1}(M)$ defined by

$$d\eta(X_1, \ldots, X_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i-1} X_i[\eta(X_1, \ldots, \hat{X}_i, \ldots, X_{q+1})]$$

$$+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{q+1})$$

where $X_1, \ldots, X_{q+1}$ are vector fields. A caret over a term means that this term is omitted. According to this definition,

$$f \in \Lambda^0(M) \Rightarrow df(X) = X(f),$$

$$\eta \in \Lambda^1(M) \Rightarrow d\eta(X, Y) = X[\eta(Y)] - Y[\eta(X)] - \eta([X, Y]).$$

Let us show that $d\eta$ is indeed a differential form. It is obvious that $d\eta(X, Y) = -d\eta(Y, X)$ and that $d\eta(X_1 + X_2, Y) = d\eta(X_1, Y) + d\eta(X_2, Y)$.

But we have to prove the $C^\infty(M)$-linearity: for any $f \in C^\infty(M)$,

$$d\eta(fX, Y) = f d\eta(X, Y).$$

According to the expression of $d\eta$,

$$d\eta(fX, Y) = fX[\eta(Y)] - Y[\eta(fX)] - \eta([fX, Y])$$

$$= fX[\eta(Y)] - Y[f\eta(X)] - \eta(f[X, Y] - Y(f)X),$$

since $[fX, Y] = dY(fX) - d(fX)Y = f[X, Y] - (df)YX$.

As $Y[f\eta(X)] = fY[\eta(X)] + \eta(X)Y(f)$, we have proved the announced result for a differential 1-form.

For $q > 1$ the proof is similar, but we have a simpler expression of the exterior differential, from which it is obvious that $d\eta$ is an exterior differential $(q + 1)$-form.

2.25. Local expression of the exterior differential. Let $(\Omega, \varphi)$ be a local chart, $x^1, \ldots, x^n$ the corresponding coordinates, $\{\partial/\partial x^i\}$ ($i = 1, \ldots, n$) the $n$ vector fields of the natural basis, and $\{dx^i\}$ the dual basis.

The differential $q$-form $\eta$ is written

$$\eta = \sum_{j_1 < \ldots < j_q} a_{j_1 \ldots j_q}(x) dx^{j_1} \wedge \cdots \wedge dx^{j_q},$$
where \( a_{j_1 \ldots j_q} \) are differentiable real-valued functions.

Since \([ \partial/\partial x^i, \partial/\partial x^j ] = 0\), by definition
\[
d\eta \left( \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_{q+1}}} \right)
= \sum_{k=1}^{q+1} (-1)^{k-1} \frac{\partial}{\partial x^{j_k}} \left[ \eta \left( \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}}, \ldots, \frac{\partial}{\partial x^{j_{q+1}}} \right) \right]
= \sum_{k=1}^{q+1} (-1)^{k-1} \frac{\partial a_{j_1 \ldots j_k \ldots j_{q+1}}}{\partial x^{j_k}}.
\]

Thus, we easily verify that
\[
d\eta = \sum_{j_1 < j_2 < \cdots < j_q} da_{j_1 \ldots j_q} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}.
\]

From this expression we see that
\[
d(\eta \wedge \xi) = d\eta \wedge \xi + (-1)^q \eta \wedge d\xi
\]
and then that \( d^2 = 0 \). On differential forms, we will speak of the differential instead of the exterior differential.

2.26. Proposition. Let \( M \) and \( W \) be differentiable manifolds and \( \Phi \) a differentiable map of \( M \) into \( W \). For any exterior differential form \( \eta \in \Lambda(W) \),
\[
d(\Phi^* \eta) = \Phi^*(d\eta).
\]

We saw this result on the functions \( f \in \Lambda^0(W) \):
\[
d(f \circ \Phi) = \Phi^*(df) \quad \text{(see Definition 2.7)}.
\]

For a differential 1-form \( \eta = df \in \Lambda^1(W) \) the proposition is trivially true since \( d^2 = 0 \). Moreover (Proposition 2.23), \( \Phi^*(\eta \wedge \xi) = \Phi^*(\eta) \wedge \Phi^*(\xi) \).

Choose local charts on \( M \) and \( W \), \( \{ x^i \} \) being the coordinate system in the considered chart on \( W \). We have (see 2.25)
\[
d\eta = \sum_{j_1 < j_2 < \cdots < j_q} da_{j_1 \ldots j_q} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q}.
\]

Thus
\[
\Phi^*(d\eta) = \sum_{j_1 < j_2 < \cdots < j_q} (\Phi^* da_{j_1 \ldots j_q}) \wedge \Phi^* dx^{j_1} \wedge \cdots \wedge \Phi^* dx^{j_q}
= \sum_{j_1 < j_2 < \cdots < j_q} d(a_{j_1 \ldots j_q} \circ \Phi) \wedge d(x^{j_1} \circ \Phi) \wedge \cdots \wedge d(x^{j_q} \circ \Phi).
\]
On the other hand,
\[
\Phi^* \eta = \sum_{j_1 < j_2 < \cdots < j_q} (a_{j_1 \cdots j_q} \circ \Phi) d(x^{j_1} \circ \Phi) \wedge \cdots \wedge d(x^{j_q} \circ \Phi),
\]
and according to 2.25
\[
d(\Phi^* \eta) = \sum_{j_1 < j_2 < \cdots < j_q} d(a_{j_1 \cdots j_q} \circ \Phi) \wedge d(x^{j_1} \circ \Phi) \wedge \cdots \wedge d(x^{j_q} \circ \Phi).
\]

**Orientable Manifolds**

2.27. Definition. A differentiable manifold is said to be orientable if there exists an atlas (in the equivalence class) all of whose changes of charts have positive Jacobian.

Given two charts of the atlas, \((\Omega, \varphi)\) and \((\theta, \psi)\), with \(\Omega \cap \theta \neq \emptyset\), denote by \(\{x^i\}\) the coordinates corresponding to \((\Omega, \varphi)\) and by \(\{y^a\}\) those corresponding to \((\theta, \psi)\). In \(\Omega \cap \theta\), let \(A^a_i = \partial y^a / \partial x^i\) and \(B^b_j = \partial x^j / \partial y^b\). According to the definition, the Jacobian matrix \(A = ((A^a_i)) \in GL(\mathbb{R}^n)^+,\) the subgroup of \(GL(\mathbb{R}^n)\) consisting of those matrices \(A\) for which \(\det A = |A| > 0\).

2.28. Theorem. A differentiable manifold \(M\) is orientable if and only if there exists a differential \(n\)-form that is everywhere nonvanishing.

Suppose \(M\) is orientable. Let \((\Omega_i, \varphi_i)\) be a locally finite atlas, all of whose changes of charts have positive Jacobian, and \(\{\alpha_i\}\) a partition of unity subordinate to the covering \(\{\Omega_i\}\). Let \(x^1, x^2, \ldots, x^n\) be the coordinates on \(\Omega_i\), and consider the differential \(n\)-form \(\omega_i = \alpha_i(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n\).

Let us verify that \(\omega = \sum_{i \in I} \omega_i\) is nowhere zero. A given point \(P\) belongs only to a finite number of \(\Omega_i\); let \(\Omega_1, \Omega_2, \ldots, \Omega_n\) be these \(\Omega_i\).

Write all \(\omega_i(P)\) in the same coordinate system \(\{x^i\}\):
\[
\omega(P) = \left[\alpha_1(P) + \sum_{i=2}^{m} \frac{\partial x^j}{\partial x^k} \bigg|_P \alpha_i(P)\right] dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.
\]
\(\omega(P)\) does not vanish, since each term in the bracket is nonnegative and some of them are strictly positive. Recall that \(\sum_{i=1}^{m} \alpha_i(P) = 1, \alpha_i(P) \geq 0\) and \(|\partial x^j / \partial x^k| > 0\).

Conversely, let \(\omega\) be a nonvanishing differential \(n\)-form, and \(A = (\Omega_i, \varphi_i)_{i \in I}\) an atlas such that all the \(\Omega_i\) are connected. From \(A\) we will construct an atlas all of whose changes of charts are positive. On \(\Omega_i\) there exists a nonvanishing function \(f_i\) such that \(\omega(x) = f_i(x) dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n\). Since \(\omega\) does not vanish and since \(\Omega_i\) is connected, \(f_i\) has a fixed sign. If \(f_i\) is positive, we keep the chart \((\Omega_i, \varphi_i)\). In that case we set \(\tilde{\varphi}_i = \varphi_i\). Otherwise, whenever \(f_j\) is negative, we consider \(\tilde{\varphi}_j\), the composition of \(\varphi_j\) with
the transformation \((x^1, x^2, \ldots, x^n) \rightarrow (-x^1, x^2, \ldots, x^n)\) of \(\mathbb{R}^n\). So from \(\mathcal{A}\) we construct an atlas \(\tilde{\mathcal{A}} = (\Omega_i, \tilde{\varphi}_i)_{i \in I}\).

Now if \(\tilde{f}_i\) is equivalent to \(f_i\) but in the chart \((\Omega_i, \tilde{\varphi}_i)\), then \(\tilde{f}_i = f_i\) if \(\varphi_i = \tilde{\varphi}_i\), \(\tilde{f}_j = -f_j\) otherwise. So \(\tilde{f}_j\) and \(f_i\) are positive. At \(x \in \Omega_i \cap \Omega_j\), denoting by \(|A|\) the determinant of the Jacobian of \(\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}\), we have \(f_j|A| = \tilde{f}_i\). Thus \(|A| > 0\) and all changes of charts of \(\tilde{\mathcal{A}}\) have positive Jacobian.

\[2.29. \text{Definition. Let } M \text{ be a connected orientable manifold. On the set of nonvanishing differential } n\text{-forms, consider the following equivalence relation: } \omega_1 \sim \omega_2 \text{ if there exists } f > 0 \text{ such that } \omega_1 = f \omega_2. \text{ There are two equivalence classes. Choosing one of them defines an orientation of } M; \text{ then } M \text{ is called oriented. There are two possible orientations of an orientable connected manifold.}\]

Some examples of orientable manifolds are the sphere, the cylinder, the torus, real projective spaces of odd dimension, the tangent bundle of any manifold, and complex manifolds.

Some examples of nonorientable manifolds are the Möbius band, the Klein bottle, and real projective spaces of even dimension (see 1.9).

We can see the Möbius band in \(\mathbb{R}^3\). Consider a rectangle \(ABCD\) in \(\mathbb{R}^2\): \([-1, 1] \times \epsilon, \epsilon > 0, A = (1, \epsilon), B = (1, -\epsilon), C = (-1, -\epsilon), D = (-1, \epsilon)\), and identify the segment \(AB\) with \(CD\). If we do this by identifying \(A\) with \(D\) and \(B\) with \(C\), we have a cylinder. If instead we identify \(A\) with \(C\) and \(B\) with \(D\), we get a Möbius band.

Let us consider the atlas \(\mathcal{A}\) with two charts, \((\Omega_1, \varphi_1), (\Omega_2, \varphi_2)\), where

\[
\Omega_1 = [-\frac{3}{4}, \frac{3}{4}] \times \epsilon, \epsilon \subset \mathbb{R}^2, \\
\varphi_1 = \text{identity}, \\
\Omega_2 = [\frac{1}{2}, 1] \times \epsilon, \epsilon \subset [-1, -\frac{1}{2}] \times \epsilon, \epsilon, \\
\varphi_2 = \text{identity on } [\frac{1}{2}, 1] \times \epsilon, \epsilon, \\
\varphi_2(x, y) = (x + 2, -y) \text{ on } [-1, -\frac{1}{2}] \times \epsilon, \epsilon, 
\]
(x, y are coordinates on \( \mathbb{R}^2 \)). Then

\[
\Omega_1 \cap \Omega_2 = \left\{ \frac{3}{4} \cdot \frac{1}{2} \cdot x \right\} - \epsilon, \epsilon \left\{ \frac{1}{2} \cdot \frac{3}{4} \cdot x \right\} - \epsilon, \epsilon \subset \mathbb{R}^2.
\]

Let \((x_1, y_1)\) and \((x_2, y_2)\) be the considered coordinate systems on \(\Omega_1\) and \(\Omega_2\) respectively.

On \(\frac{3}{4} \cdot \frac{1}{2} \cdot x\) the change of coordinates is \(x_2 = x_1, y_2 = y_1\), and on \(-\frac{3}{4} \cdot \frac{1}{2} \cdot x\) it is \(x_2 = x_1 + 2, y_2 = -y_1\). On the first open set the change of coordinate chart has positive Jacobian, but on the second it has negative Jacobian.

In the proof of Theorem 2.28 we saw that, if a manifold is orientable, from an atlas \(\mathcal{A}\) we can construct an atlas \(\mathcal{A}\) all of whose changes of charts are positive only by the eventual change of coordinate \(x_2 \rightarrow -x_2\). Here if we do that, we always have one positive Jacobian and one negative Jacobian on the two open sets of \(\Omega_1 \cap \Omega_2\). Thus the Möbius band is not orientable.

The Klein bottle is the compact version of the Möbius band. It is in \(\mathbb{R}^4\). Consider a cylinder with end circles \(C_1\) and \(C_2\). Let \(ABCD\) be four points (in that order) on \(C_1\), and \(A'B'C'D'\) four points (again, in order) on \(C_2\). We identify \(C_1\) and \(C_2\). We get a torus if we identify \(A\) with \(A'\), \(B\) with \(B'\), \(C\) with \(C'\), and \(D\) with \(D'\). On the other hand, we get a Klein bottle if we identify \(A\) with \(A'\), \(B\) with \(D'\), \(C\) with \(C'\), and \(D\) with \(B'\).

The cylinder is a two-sheeted covering manifold of the Möbius band. The torus is a two-sheeted covering manifold of the Klein bottle.

2.30. Theorem. If \(M\) is nonorientable, \(M\) has a two-sheeted orientable covering manifold \(\tilde{M}\). If \(M\) is simply connected, then \(M\) is orientable.

For the proof see Narasimhan [10].

\(M\) is said to be simply connected if any closed curve in \(M\) may be reduced to one point by a continuous deformation. More precisely,

2.31. Definition. Let \(C\) be a circle, \(f\) and \(g\) two continuous (respectively differentiable) maps of \(C\) into a manifold \(M\). Let \(f(C)\) and \(g(C)\) be closed (respectively differentiable) curves of \(M\). \(f(C)\) is said to be homotopic to \(g(C)\) if there exists a continuous map \(F(s, t)\) of \(C \times [0, 1]\) into \(M\) such that \(F(x, 0) = f(x)\) and \(F(x, 1) = g(x)\). If \(f\) and \(g\) are \(C^1\), we require that \(F(x, t)\) be \(C^1\) in \(x\) on \(C \times [0, 1]\).

2.32. Definition. The manifold \(M\) is said to be simply connected if any closed curve \(f(C)\) in \(M\) is homotopic to one point; that is, there exists \(F(x, t)\) as above with \(g(C)\) reduced to one point. \(f(C)\) homotopic to \(g(C)\) is an equivalence relation. The equivalence classes are called homotopy classes.

2.33. Definition. Let \(M_n\) be a differentiable orientable manifold. We define the integral of \(\omega\), a differential \(n\)-form with compact support, as follows:
2. Tangent Space

Let \((\Omega_i, \varphi_i)_{i \in I}\) be an atlas compatible with the orientation chosen, and \(\{\alpha_i\}_{i \in I}\) a partition of unity subordinate to the covering \(\{\Omega_i\}_{i \in I}\). On \(\Omega_i\), \(\omega\) is equal to \(f_i(x)dx_1^1 \wedge dx_1^2 \wedge \cdots dx_1^n\). By definition,

\[
\int_M \omega = \sum_{i \in I} \int_{\varphi_i(\Omega_i)} [\alpha_i(x)f_i(x)] \circ \varphi_i^{-1} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.
\]

One may verify that the definition makes sense. The integral does not depend on the partition of unity, and the sum is finite.

Manifolds with Boundary

2.34. Let \(E\) be the half-space \(x_1 > 0\) of \(\mathbb{R}^n\), \(x^1\) being the first coordinate of \(\mathbb{R}^n\). Consider \(\overline{E} \subset \mathbb{R}^n\) with the induced topology. We identify the hyperplane \(\Pi\) of \(\mathbb{R}^n\), \(x^1 = 0\), with \(\mathbb{R}^{n-1}\). Letting \(\Omega\) and \(\theta\) be two open sets of \(\overline{E}\), and \(\varphi : \Omega \to \theta\) a homeomorphism, it is possible to prove that the restriction of \(\varphi\) to \(\Omega \cap \Pi\) is a homeomorphism of \(\Omega \cap \Pi\) onto \(\theta \cap \Pi\). So the boundary \(\Pi\) of the manifold with boundary \(\overline{E}\) is preserved by homeomorphism. \(\overline{E}\) is the standard manifold with boundary, as \(\mathbb{R}^n\) is the standard manifold.

2.35. Definition. A separated topological space \(M_n\) is a manifold with boundary if each point of \(M_n\) has a neighbourhood homeomorphic to an open set of \(\overline{E}\).

The points of \(M_n\) which have a neighbourhood homeomorphic to \(\mathbb{R}^n\) are called interior points. They form the interior of \(M_n\). The other points are called boundary points. We denote the set of boundary points by \(\partial M\), and call it the boundary of \(M\).

As in 1.6, we define a \(C^k\)-differentiable manifold with boundary. By definition, a function is \(C^k\)-differentiable on \(\overline{E}\) if it is the restriction to \(\overline{E}\) of a \(C^k\)-differentiable function on \(\mathbb{R}^n\).

2.36. Theorem. Let \(M_n\) be a (\(C^k\)-differentiable) manifold with boundary. If \(\partial M\) is not empty, then \(\partial M\) is a (\(C^k\)-differentiable) manifold of dimension \(n - 1\), without boundary: \(\partial(\partial M) = \emptyset\).
Proof. If \( Q \in \partial M \), there exists a neighbourhood \( \Omega \) of \( Q \) homeomorphic by \( \varphi \) to an open set \( \Theta \subset \overline{E} \). The restriction \( \hat{\varphi} \) of \( \varphi \) to \( \hat{\Omega} = \Omega \cap \partial M \) is a homeomorphism of the neighbourhood \( \hat{\Omega} \) of \( Q \in \partial M \) onto an open set \( \hat{\Theta} = \Theta \cap \Pi \) of \( \mathbb{R}^{n-1} \). Thus \( \partial M \) is a manifold (without boundary) of dimension \( n - 1 \) (Definition 1.1). If \( M \) is \( C^k \)-differentiable, let \((\Omega_i, \varphi_i)_{i \in I}\) be a \( C^k \)-atlas. Clearly, \((\hat{\Omega}_i, \hat{\varphi}_i)_{i \in I}\) turns out to be a \( C^k \)-atlas for \( \partial M \).

2.37. Theorem. If \( M \) is a \( C^k \)-differentiable oriented manifold with boundary, then \( \partial M \) is orientable. An orientation of \( M \) induces a natural orientation of \( \partial M \).

Proof. Let \((\Omega_j, \varphi_j)_{j \in I}\) be an admissible atlas with the orientation of \( M \), and \((\hat{\Omega}_j, \hat{\varphi}_j)_{j \in I}\) the corresponding atlas of \( \partial M \), as above. Let \( i: \partial M \to M \) be the canonical imbedding of \( \partial M \) into \( M \). We identify \( Q \) with \( i(Q) \), and \( X \in T_Q(\partial M) \) with \( i_*(X) \in T_Q(M) \). Given \( Q \in \partial M \), pick \( e_1 \in T_Q(M) \), \( e_1 \notin T_Q(\partial M) \), \( e_1 \) being oriented to the outside, namely, \( e_1(f) \geq 0 \) for all differentiable functions on a neighbourhood of \( Q \) which satisfy \( f \leq 0 \) in \( M \) and \( f(Q) = 0 \). We choose a basis \( \{e_2, e_3, \ldots, e_n\} \) of \( T_Q(\partial M) \) such that the basis \( \{e_1, e_2, e_3, \ldots, e_n\} \) of \( T_Q(N) \) belongs to the positive orientation given on \( M \). Then \( \{e_2, e_3, \ldots, e_n\} \) is a positive basis for \( T_Q(\partial M) \).

This procedure defines a canonical orientation on \( \partial M \), as one can see.

2.38. Stokes' Formula. Let \( M \) be a \( C^k \)-differentiable oriented compact manifold with boundary, and \( \omega \) a differential \((n-1)\)-form on \( M \); then

\[
\int_M d\omega = \int_{\partial M} \omega,
\]

where \( \partial M \) is oriented according to the previous theorem. For convenience we have written \( \int_{\partial M} \omega \) instead of \( \int_{\partial M} i^* \omega \) (for the definition of \( i^* \) see 2.7 and 2.25).

Proof. Let \((\Omega_i, \varphi_i)_{i \in I}\) be a finite atlas compatible with the orientation of \( M \); such an atlas exists, because \( M \) is compact. Set \( \Theta_i = \varphi_i(\Omega_i) \). Consider \( \{\alpha_i\} \), a \( C^k \)-partition of unity subordinate to \( \{\Omega_i\} \). By definition,

\[
\int_M d\omega = \sum_{i \in I} \int_{\Theta_i} d(\alpha_i \omega).
\]

Thus we have only to prove that

\[
\int_{\Theta_i} d(\alpha_i \omega) = \int_{\Theta_i} (\alpha_i \omega).
\]
We recall that $\tilde{\Omega}_i = \Omega_i \cap \partial M$ and we have set $\tilde{\Theta}_i = \varphi_i(\tilde{\Omega}_i) = \Theta_i \cap \Pi$. In $(\Omega_i, \varphi_i)$ we have

$$\alpha_i \omega = \sum_{j=1}^{n} f_j(x) dx^1 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx^n,$$

where the $f_j(x)$ are $C^k$-differentiable functions with compact support included in $\varphi_i(\Omega_i)$; $\dot{d}x_j$ means this term is missing. Now,

$$d(\alpha_i \omega) = \sum_{j=1}^{n} df_j(x) \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx_j \wedge \cdots \wedge dx^n$$

$$= \left[ \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial f_j(x)}{\partial x^j} \right] dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

by 2.25. According to Fubini's theorem,

$$\int_{\tilde{\Theta}_i} d(\alpha_i \omega) = \int_{\tilde{\Theta}_i} f_1(x) dx^2 \wedge dx^3 \wedge \cdots \wedge dx^n = \int_{\tilde{\Theta}_i} j^*(\alpha_i \omega),$$

where $j$ is the inclusion $\Pi \to \overline{E}$. Indeed,

$$j^*(\alpha_i \omega) \left( \frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n} \right) = \alpha_i \omega \left( j_* \frac{\partial}{\partial x^2}, \cdots, j_* \frac{\partial}{\partial x^n} \right) = f_1(x),$$

and we identify $j_* \partial/\partial x^k$ with $\partial/\partial x^k$. Observe that

$$\omega \xrightarrow{i^*} i^* \omega \quad (\phi^{-1})^* \downarrow \quad (\phi^{-1})^* \omega \xrightarrow{j^*} \Lambda^{n-1}(\Pi)$$

So, we have $(\phi^{-1})^* \circ i^* = j^* \circ (\phi^{-1})^*$.

**Exercises and Problems**

2.39. Exercise. Let $M$ and $W$ be two differentiable manifolds, and $f$ a diffeomorphism of $M$ onto $W$. Let $P \in M$, and set $Q = f(P)$. Consider $Y \in T_Q(W)$ and $\tilde{\gamma} \in \Lambda^q(W)$, $q > 1$. Express $f^* i(Y) \tilde{\gamma}$ in terms of $\gamma = f^* \tilde{\gamma}$.

2.40. Exercise. Let $\Omega$ be a bounded connected open set of $\mathbb{R}^3$ such that $\overline{\Omega}$ is a differentiable manifold with boundary. On $\mathbb{R}^3$, endowed with an orthonormal coordinate system $(x^1, x^2, x^3)$, we consider a vector field $\overrightarrow{X}$ of components $X^i(x)$:

$$\overrightarrow{X} (x) = \sum_{i=1}^{3} X^i(x) \frac{\partial}{\partial x^i}.$$
Is the boundary $\partial \Omega$ orientable? At $P \in \partial \Omega$, let $\overrightarrow{v}(P)$ be the normal unit vector of $\partial \Omega$ oriented to the outside of $\Omega$.

Prove the well-known formula

$$\int_{\Omega} \sum_{i=1}^{3} \frac{\partial X^i}{\partial x^i} dE = \int_{\partial \Omega} \overrightarrow{v} \cdot \overrightarrow{X} d\sigma,$$

where $dE$ is the volume element on $\mathbb{R}^3$, $d\sigma$ the area element on $\partial \Omega$, and $\overrightarrow{v} \cdot \overrightarrow{X}$ the scalar product of $\overrightarrow{v}$ and $\overrightarrow{X}$. Hint. Proceed as follows:

a) Find a differential 2-form $\omega$ on $\mathbb{R}^3$ such that

$$d\omega = \sum_{i=1}^{3} \frac{\partial X^i}{\partial x^i} dx^1 \wedge dx^2 \wedge dx^3.$$

b) Let $(u, v)$ be a coordinate system on a neighbourhood $U \subset \partial \Omega$ of $P$, orthonormal at $P$ and such that $(\overrightarrow{v}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ is a positive basis in $T_p(\mathbb{R}^3)$. We set

$$\overrightarrow{X} = \frac{\partial}{\partial u} \text{ and } \overrightarrow{Y} = \frac{\partial}{\partial v}.$$

Compute $\omega(\frac{\partial}{\partial u}, \frac{\partial}{\partial v})$ in terms of the components of $\overrightarrow{X}$, $\frac{\partial}{\partial u}$, and $\frac{\partial}{\partial v}$.

c) Use Stokes' formula.

2.41. Exercise. Consider on $\mathbb{R}^3$, endowed with a coordinate system $(x, y, z)$, the following three vector fields:

$$X = \frac{1}{2}(1 + x^2 - y^2 - z^2) \frac{\partial}{\partial x} + (xy - z) \frac{\partial}{\partial y} + (xz + y) \frac{\partial}{\partial z},$$

$$Y = (xy + z) \frac{\partial}{\partial x} + \frac{1}{2}(1 - x^2 + y^2 - z^2) \frac{\partial}{\partial y} + (yz - x) \frac{\partial}{\partial z},$$

$$Z = (xz - y) \frac{\partial}{\partial x} + (yz + x) \frac{\partial}{\partial y} + \frac{1}{2}(1 - x^2 - y^2 + z^2) \frac{\partial}{\partial z}.$$

a) Verify that the three vectors $X(m)$, $Y(m)$, $Z(m)$ form an orthogonal basis of $\mathbb{R}^3$ at each point $m \in \mathbb{R}^3$ (the components of the Euclidean metric are in the coordinate system $\delta_{ij} = \delta_i^j$).

b) Compute the three Lie brackets $[X, Y]$, $[Y, Z]$, $[Z, X]$ and express them in the basis $(X, Y, Z)$.

c) Let $\Omega = \mathbb{R}^3 - \{0\}$ and consider the map $\varphi$ of $\Omega$ into $\Omega$, $(x, y, z) \rightarrow (u, v, w)$, defined by

$$u = \frac{x}{(x^2 + y^2 + z^2)},$$
2. Tangent Space

\[ v = \frac{y}{x^2 + y^2 + z^2}, \]
\[ w = \frac{z}{x^2 + y^2 + z^2}. \]

The restrictions of \( X, Y, Z \) to \( \Omega \) are still denoted \( X, Y, Z \). Verify that \( \varphi \) is a diffeomorphism, and compute \( \varphi_*X, \varphi_*Y \) and \( \varphi_*Z \).

d) Using an atlas with two charts on \( S_3 \), deduce from c) the existence of three vector fields on \( S_3 \) forming a basis of the tangent space at each point of \( S_3 \). Notice that \( \varphi \) is the diffeomorphism of change of charts for an atlas with two charts on \( S_3 \) constructed by stereographic projection.

2.42. Exercise. Let \( W \) be the set of real \( 3 \times 3 \) matrices whose determinant is equal to 1.

a) Exhibit a differentiable manifold structure on \( W \). Show that \( W \) is a hypersurface of \( \mathbb{R}^n \), and specify \( n \).

b) Identify to a set of matrices the tangent space \( T_I(W) \) of \( W \) at \( I \), the identity matrix.

2.43. Problem. In \( \mathbb{R}^3 \), we consider a compact and connected differentiable submanifold \( M \) of dimension 2.

a) Show that \( \Omega = \mathbb{R}^3 \setminus M \) has at most two connected components.

b) We admit that \( \Omega \) has at least two connected components. Prove that one of them is bounded. We call it \( W \).

c) Construct on \( M \) a continuous field \( \nu(P) \) of unit vectors in \( \mathbb{R}^3 \) such that \( \nu(P) \) is orthogonal to \( T_P(M) \) at each point \( P \in M \). The norm of the vectors comes from the scalar product \( \langle ., . \rangle \) defined by the Euclidean metric on \( \mathbb{R}^3 \) endowed with a coordinate system \( \{x^i\}, i = 1, 2, 3 \).

d) Deduce that \( M \) is orientable.

e) On \( \mathbb{R}^3 \), consider the differential 3-form \( \omega = dx^1 \wedge dx^2 \wedge dx^3 \) and a vector field \( X \). Verify that

\[ d[i(X)\omega] = (\text{div} X)\omega, \quad \text{where} \quad \text{div} X = \sum_{k=1}^{3} \frac{\partial X_k}{\partial x^k}. \]

f) Prove that

\[ \int_W (\text{div} X)\omega = \int_M \langle X, \nu \rangle j^*[i(\nu)\omega] \]

for an orientation on \( M \). Here \( j \) is the inclusion \( M \subset W \).

g) Let \( f \) be a \( C^2 \) function defined on \( \mathbb{R}^3 \). We suppose that \( f \) satisfies \( \sum_{i=1}^{3} \partial_{x_i} f = 0 \) on \( W \).

If \( f|_M \equiv 0 \), or if \( \partial_{\nu} f \equiv 0 \) on \( M \), prove that \( f \) is constant on \( W \).
h) When $M$ is the set of zeros of a $C^1$ function on $\mathbb{R}^3$ such that $\mathbb{R}^3 \ni x \to f(x)$ is of rank 1 at any point $x \in M$, show that $\Omega$ has at least two connected components.

2.44. Problem. In this problem $\mathbb{R}^n$ is endowed with the Euclidean metric $E$, $\{x^i\}$ is a coordinate system denoted $(x, y, z)$ on $\mathbb{R}^3$, and $S_{n-1}$ is the set of the points of $\mathbb{R}^n$ satisfying $\sum_{i=1}^{n} (x^i)^2 = 1$. $\Psi$ is the inclusion $S_{n-1} \subset \mathbb{R}^n$.

a) On $\mathbb{R}^3 - \{0\}$, consider the differential 2-form

$$\omega = (xdy \wedge dz - ydx \wedge dz + zdx \wedge dy)(x^2 + y^2 + z^2)^{-\frac{3}{2}}.$$ 

Verify that $\omega$ is closed.

b) Compute $\int_{S_2} \Psi^* \omega$. For lack of a better method, one can use spherical coordinates.

c) Is $\omega$ homologous to zero on $\mathbb{R}^3 - \{0\}$? (See 5.18 for the definition.)

d) On $\mathbb{R}^n - \{0\}$, consider the differential 1-form

$$\alpha = \left( \sum_{i=1}^{n} x^i dx^i \right)^{-\frac{n}{2}} \left( \sum_{j=1}^{n} (x^j)^2 \right)^{-\frac{n}{2}}.$$ 

Compute $\star \alpha$ (the adjoint of $\alpha$, see 5.16 for the definition; here $\eta_{12..n} = 1$) and prove that $\star \alpha$ is closed.

e) What is the value of $\int_{S_{n-1}} \Psi^*(\star \alpha)$? Is $\star \alpha$ homologous to zero on $\mathbb{R}^n - \{0\}$?

2.45. Exercise. Let $O(n)$ be the set of the $n \times n$ matrices $M$ such that $^tM = M = I$, the identity matrix.

a) Show that $O(n)$ is a manifold. What is its dimension? Hint. If you want, first consider the problem in a neighbourhood of $I$.

b) Identify $T_I(O(n))$ with a set $S$ of $n \times n$ matrices.

c) Show that if $A \in S$ and $M \in O(n)$, then $MA \in T_M(O(n))$. Verify that the map $\Psi : (M, A) \to (M, MA)$ is a diffeomorphism of $O(n) \times S$ onto $T(O(n))$.

d) Deduce from this result that $O(n)$ is parallelisable.

2.46. Problem. In this problem $\omega$ is a $C^\infty$ differential 1-form on $\mathbb{R}^{n+1}$ which is not zero at $0 \in \mathbb{R}^{n+1}$.

a) Show that if there exist two $C^\infty$ functions $f$ and $g$, defined on a neighbourhood $V$ of $0 \in \mathbb{R}^{n+1}$ with $f(0) \neq 0$, such that $\omega = f dg$ on $V$, then there exists a differential 1-form $\theta$ in a neighbourhood of $0 \in \mathbb{R}^{n+1}$ such that $d\omega = \theta \wedge \omega$. 
b) Exhibit an expression of $\theta$ if $\omega = yzdx + xzdy + dz$, $(x, y, z)$ being a coordinate system on $\mathbb{R}^3$. Show that we can choose $f = e^{-x^2}$. Compute $g$.

c) If $\omega = dz - ydx - dy$, do $f$ and $g$ exist?

d) Next, put $\omega$ in the form $\omega = f dg$. In the general case, show that the problem may be reduced to the case where, in a neighbourhood of 0,

$$\omega = dz - \sum_{i=1}^{n} A_i(x, z)dx^i.$$ 

Here $z \in \mathbb{R}$, $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$, and the $A_i$ are $C^\infty$ functions.

e) $a = (a^1, a^2, \ldots, a^n) \in \mathbb{R}^n$ and $c \in \mathbb{R}$ being given, consider the differential equation

$$\frac{dz}{dt} = \sum_{i=1}^{n} A_i(at, z)a^i, \quad z(0) = c.$$ 

Show that there exists a $C^\infty$ map $F : (t, a, v) \rightarrow F(t, a, v)$ of $I \times W \times J$ into $\mathbb{R}$ with $I, W, J$ neighbourhoods respectively of $0 \in \mathbb{R}$, $0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$, such that

$$\frac{\partial F}{\partial t} = \sum_{i=1}^{n} A_i(at, F)a^i, \quad F(0, a, c) = c.$$ 

Verify that $F(t, a, v) = F(1, at, v)$ if one of these terms exists.

f) Prove that $u$ and $v$ defined by $u = x$ and $F(1, u, v) = z$ form a coordinate system in a neighbourhood of $(0, c) \in \mathbb{R}^n \times \mathbb{R}$. Hint. Show that $(\partial F/\partial v)(t, a, v)$ does not vanish in a neighbourhood of $(0, 0, c)$. In this chart $\omega = \sum_{i=1}^{n} P_i(u, v)du^i + B(u, v)dv$. Show that

$$\sum_{i=1}^{n} P_i(at, v)a^i = 0.$$ 

g) Let $\Psi$ be the map of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$ into $\mathbb{R}^n \times \mathbb{R}$ defined by

$$\Psi(t, a, v) = (ta, v) = (u, v).$$ 

Compute $\Psi^*\omega$. Denote by $R_i(t, a, v)$ the coefficient of $da^i$ in $\Psi^*\omega$.

h) We suppose that $d\omega = \theta \wedge \omega$, $\theta$ being a differential 1-form in a neighbourhood of $0 \in \mathbb{R}^{n+1}$. Show that $\partial R_i/\partial t = H R_i$, with $H(t, a, v)$ the coefficient of $dt$ in $\Psi^*\theta$.

Deduce from this result that there exists a neighbourhood of $0 \in \mathbb{R}^{n+1}$ where $\omega = Bdv$.

2.47. Exercise. Let $M$ be a $C^\infty$ differentiable manifold of dimension $n$. Consider its cotangent bundle $T^*(M)$, and denote by $\Pi$ the canonical projection $T^*(M) \rightarrow M$. 

a) \((\theta, \varphi)\) being a local chart on \(M\) and \(\{\xi^i\}\) the corresponding coordinate system, on \(\Pi^{-1}(\theta)\) consider the coordinate system \((x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n)\) with \(x^i = \xi^i \circ \Pi\) and \(y^i\) the components of the 1-form in the basis \(\{\mathrm{d}\xi^i\}\) \((i = 1, 2, \ldots, n)\). Let \(\alpha \in T^*_2(M)\), and define a linear form on \(T_\alpha(T^*(M))\) by

\[
T_\alpha(T^*(M)) \ni u \longrightarrow \langle \Pi^* u, \alpha \rangle = \langle u, \Pi^* \alpha \rangle.
\]

Show that \(\alpha \longrightarrow \Pi^* \alpha\) defines a differential 1-form \(\omega\) on \(M\). What is its expression?

b) Compute \(\Omega = \omega \wedge \omega \wedge \cdots \wedge \omega\). Deduce from the result that \(T^*(M)\) is orientable.

2.48. Exercise. Prove that for any \(C^\infty\) differentiable manifold \(M\) the tangent space \(T(M)\) is orientable.

2.49. Problem.

a) Describe an atlas of the projective space \(P_2(\mathbb{R})\) with three local charts \((\Omega_i, \phi_i), i = 1, 2, 3\).

b) Compute the changes of charts, and deduce that \(P_2\) is not orientable.

c) Consider the cylinder \(H = [-1, 1] \times C\), where \(C\) is the circle quotient of \(\mathbb{R}\) by the equivalence relation in \(\mathbb{R}: \theta \sim \theta + 2k\pi\) \((k \in \mathbb{Z})\). Verify that the Möbius band \(M\) may be identified with the quotient of \(H\) by the equivalence relation in \(H:\ (t, \theta) \sim (-t, \theta + \pi)\). Prove that the boundary of \(M\) is diffeomorphic to a circle \(C : C \ni \theta \longrightarrow \phi(\theta) \in \partial M\).

d) Let \((r, \omega)\) be a polar coordinate system on the disk

\[
D = \{x \in \mathbb{R}^2 | |x| \leq 1\}.
\]

Show that \(P_2(\mathbb{R})\) may be identified with the quotient of \(D \cup M\) by the equivalence relation: \((1, \theta) \in D\) is equivalent to \(\phi(\theta) \in M\).

2.50. Problem. On a \(C^\infty\) differentiable manifold \(M\) of dimension \(2n\), we suppose that there exists a closed 2-form \(\Omega\) of rank \(2n\). That is to say that \(\Omega \wedge \Omega \wedge \cdots \wedge \Omega \neq 0\) everywhere on \(M\).

a) To a \(C^\infty\) vector field \(X \in \Gamma(M)\) we associate the 1-form \(\omega_X = i(X)\Omega\). Verify that \(\omega_X\) is closed if and only if \(\mathcal{L}_X \Omega = 0\).

b) Show that the map \(h: X \longrightarrow \omega_X\) is an isomorphism of \(\Gamma(M)\) on \(\wedge^1(M)\).

c) \(\alpha\) and \(\beta\) being two 1-forms, we set \((\alpha, \beta) = h([X_\alpha, X_\beta])\) with \(X_\alpha = h^{-1}(\alpha)\) and \(X_\beta = h^{-1}(\beta)\). If \(\alpha\) is closed, prove that \(\mathcal{L}_X (\alpha, \beta) = (\alpha, \beta)\). Deduce that \((\alpha, \beta)\) is homologous to zero if \(\alpha\) and \(\beta\) are closed.
d) Let $f, g$ be two $C^\infty$ functions on $M$, and set

$$(f, g) = \Omega(X_{dg}, X_d).$$

Show that $(df, dg) = d(f, g)$. Deduce that $f$ is constant along the integral curves of $X_{dg}$ if and only if $g$ is constant along the integral curves of $X_d$.

e) Assume that locally there exists a coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ such that $\Omega = \sum_{i=1}^n dx^i \wedge dy^i$. Compute the local expression of $(f, g)$ in this coordinate system.

2.51. Exercise.

a) Let $f$ be a differentiable map of $\mathbb{R}^n$ into $\mathbb{R}$ of maximal rank everywhere. Show that $f^{-1}(0)$ is an orientable manifold.

b) Prove that the manifold that is the product of two $C^\infty$ differentiable manifolds $M$ and $W$ is orientable if and only if $M$ and $W$ are orientable.

2.52. Exercise. Let $\omega$ be a differential 1-form on a $C^\infty$ differentiable manifold $M$ of dimension $2n + 1$. When $\omega \wedge d\omega \wedge \cdots \wedge d\omega$ is never zero on $M$, we say that $\omega$ is a $C$-form.

a) Verify that $\omega_0 = dx^{2n+1} + \sum_{i=1}^n x^{2i-1} dx^{2i}$ is a $C$-form on $\mathbb{R}^{2n+1}$ with coordinates $\{x^j\} (j = 1, 2, \ldots, 2n + 1)$.

b) Let $(\theta_k, \varphi_k)_{k \in K}$ be an atlas and $\gamma$ a differential 1-form such that on each $\theta_k$ we have $\gamma = f_k \varphi_k^* \omega_0$, with $f_k$ a differentiable function which does not vanish. Show that $\gamma$ is a $C$-form.

c) Conversely, $\gamma$ being a $C$-form, find an atlas $(\theta_k, \varphi_k)_{k \in K}$ such that $\gamma = f_k \varphi_k^* \omega_0$ on each $\theta_k$. For simplicity, do the proof when $n = 3$.

Solutions to Exercises and Problems

Solution to Exercise 2.39.

$f^*i(Y)\gamma$ is a $(q - 1)$-form on $M$. Let $X_i (1 \leq i \leq q - 1)$ be $q - 1$ vectors in $T_P(M)$. We have

$$(f^*i(Y))\gamma(X_1, X_2, \ldots, X_{q-1}) = i(Y)\gamma(Y_1, \ldots, Y_{q-1}) \quad \text{if} \quad Y_i = f_* X_i,$$

$$i(Y)\gamma(Y_1, \ldots, Y_{q-1}) = \gamma(Y, Y_1, Y_2, \ldots, Y_{q-1}) = f^*\gamma(X, X_1, X_2, \ldots, X_{q-1}).$$

Thus $f^*i(Y)\gamma = i(X)\gamma$ with $X = f^{-1}_* Y$.

Solution to Exercise 2.40.

$\Omega \subset \mathbb{R}^n$ is orientable, thus $\partial \Omega$ is orientable.
a) Set \( \omega = X^1 dx^2 \wedge dx^3 + X^2 dx^3 \wedge dx^1 + X^3 dx^1 \wedge dx^2 \), and
\[
dw = \sum_{i=1}^{3} \partial_i X^i dx^1 \wedge dx^2 \wedge dx^3.
\]
b) \( \omega(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = X^1(u^2 v^3 - u^3 v^2) + X^2(u^3 v^1 - v^3 u^1) + X^3(u^1 v^2 - u^2 v^1). \)
c) Stokes' formula gives
\[
\int_{\Omega} \sum_{j=1}^{3} \partial_j X^j dE = \int_{\partial \Omega} i^* \omega,
\]
where \( i \) is the inclusion \( \partial \Omega \to \Omega \). Let us compute \( i^* \omega \) at \( P \):
\[
i^* \omega = \omega \left( \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right) du \wedge dv = \overrightarrow{X} \cdot \nu \, d\sigma.
\]
Indeed, \( du \wedge dv \) is the area element and
\[
\nu = \left( u^2 v^3 - u^3 v^2 \right) \frac{\partial}{\partial x} + \left( u^3 v^1 - v^3 u^1 \right) \frac{\partial}{\partial y} + \left( u^1 v^2 - u^2 v^1 \right) \frac{\partial}{\partial z}.
\]
We can verify that \( \nu \) is orthogonal to \( \frac{\partial}{\partial u} \) and to \( \frac{\partial}{\partial v} \), and that \( \| \nu \| = 1 \).

Solution to Exercise 2.41.

a) Transposing \( x \to y \to z \to x \), we obtain \( X \to Y \to Z \to X \). Thus if we prove that \( X \) does not vanish and that \( X \) is orthogonal to \( Y \), the three vectors form an orthogonal basis of \( \mathbb{R}^3 \) at each point \( m \in \mathbb{R}^3 \). \( X \) does not vanish. Indeed, in that case \( z = xy, y = -xz \) and \( 1 + x^2 = y^2 + z^2 \); thus \( z = -x^2z \) and \( y = -x^2y \). We would have \( y = z = 0 \), which is impossible since \( y^2 + z^2 \geq 1 \). We verify that the scalar product \( \langle X, Y \rangle = 0 \).

b) A straightforward computation yields \( [X, Y] = -2Z \). Thus \( [Y, Z] = -2X \) and \( [Z, X] = -2Y \).

c) On \( \Omega \), \( \varphi \circ \varphi \) = Identity; thus \( \varphi \) is invertible. It is a diffeomorphism since \( \varphi \) and \( \varphi^{-1} \) are differentiable. Using the fact that \( \varphi_* = D\varphi \), a computation leads to
\[
\varphi_* X = -\frac{1}{2} (1 + u^2 - v^2 - w^2) \frac{\partial}{\partial u} - (w + uv) \frac{\partial}{\partial v} - (uw - v) \frac{\partial}{\partial w}.
\]
We obtain \( \varphi_* Y \) and \( \varphi_* Z \) by transposing.

d) We saw in 1.8 that \( \varphi \) is the diffeomorphism of change of charts for an atlas with two charts on \( S_3 \) constructed by stereographic projection from the north and south poles \( P \) and \( \tilde{P} \), respectively.

\( \varphi_* X \) extends into a vector field \( U \) on \( \mathbb{R}^3 \), and \( \varphi_* X \) at zero is equal to \( -\frac{1}{2} \frac{\partial}{\partial u} \). Likewise for \( \varphi_* Y \) and \( \varphi_* Z \) in \( V \) and \( W \).
Thus the vector field on $S_3$ equal to $X$ in one chart (on $S_3 - \{P\}$ for instance) extends by $U$ in the other chart.

**Solution to Exercise 2.42.**

a) The set $T(3, 3)$ of real $3 \times 3$ matrices $M$ is in bijection with $\mathbb{R}^9$. If $M = ((a_{ij}))$, $1 \leq i, j \leq 3$, and $\{x^k\}$ is a coordinate system on $\mathbb{R}^9$, the bijection $\Psi: M \rightarrow x = \{x^k\}$ may be defined by $x^k = a_{ij}$ with $k = 3(i - 1) + j$. $W$ is the subset of $T(3, 3)$ such that $\det |M| = 1$.

The map $\Psi: T(3, 3) \rightarrow \det |M|$ of $\mathbb{R}^9$ in $\mathbb{R}$ is of rank 1 on $W$. Indeed, $\partial \Psi / \partial a_{ij} = M_{ij}$, the minor of $a_{ij}$ in $\det |M|$. Now all minors cannot vanish on $W$, since $\det |M| = \sum_{j=1}^{3} a_{1j}M_{1j} = 1$ on $W$.

Thus according to Theorem 1.19, $W$ is a submanifold of dimension 8 in $\mathbb{R}^9$.

b) Let $t \rightarrow \varphi(t)$ be a map of a neighbourhood of zero in $\mathbb{R}$ into $W$ such that $\varphi(0) = I$.

If $\varphi(t) = ((a_{ij}(t)))$, then $a_{ij}(0) = \delta^i_j$, the Kronecker symbol. Writing $\frac{d}{dt} \det |\varphi(t)| = 0$ leads to $\sum_{i=1}^{3} a'_{ij}(0) = 0$, and the tangent vector to the curve $t \rightarrow \varphi(t)$ at $t = 0$ is the matrix $((a'_{ij}(0)))$. Thus the tangent space $T_I(W)$ may be identified with the set of real $3 \times 3$ matrices of zero trace.

**Solution to Problem 2.43.**

a) At $P \in M$, there is a local chart $(V, \varphi)$ of $\mathbb{R}^3$ ($\varphi(V)$ a ball in $\mathbb{R}^3$) such that $\varphi(V \cap M)$ is a disk $D$ of $\mathbb{R}^2$. $V \cap \Omega$ has two connected components $\theta_1$ and $\theta_2$. $\theta_1$ is included in a connected component $W$ of $\Omega$. Then $\partial W \cap V = \varphi^{-1}(D)$ is an open set of $M$.

Thus $\partial W \cap M$ is an open set in $M$, and it is nonempty. But it is also compact ($\partial W$ is closed and included in $M$, which is compact). Hence $\partial W = M$, since $M$ is connected. At the most there are two connected components: one contains $\theta_1$ and the other contains $\theta_2$. 
b) $M$ is compact, so $M$ is included in a ball $B$. $\mathbb{R}^3 \setminus B$ is connected; thus it is included in a connected component of $\Omega$, while $W$ is included in $B$.

c) At each point $P$, we denote by $\theta_2$ the open set which belongs to the connected component of $\Omega$ which is not bounded. We choose the unit vector $\nu(P)$ orthogonal to $T_P(M)$ oriented to $\theta_2$. It is well defined and continuous.

d) On each local chart on $M$ (whose coordinates are $\xi^1, \xi^2$) we arrange that $(\nu, \partial/\partial \xi^1, \partial/\partial \xi^2)$ is a positive basis in $\mathbb{R}^3$ (this means replace $\xi^1$ by $-\xi^1$, if necessary).

Thus we exhibit an atlas for $M$, all of whose changes of charts have positive Jacobian.

e) We have

$$
i(X)\omega = X^1 dx^2 \wedge dx^3 - X^2 dx^1 \wedge dx^3 + X^3 dx^1 \wedge dx^2,$$

$$d[i(X)\omega] = \sum_{k=1}^{3} \frac{\partial X^k}{\partial x^k} \omega.$$

f) According to Stokes' formula, for the canonical orientation on $M$ (see 2.37) we have

$$\int_W \text{div} \, X \omega = \int_M j^* [i(X)\omega],$$

$$[i(X)\omega](e_1, e_2) = \omega(X, e_1, e_2) = \omega((X, \nu)\nu, e_1, e_2)$$

$$= \langle X, \nu \rangle [i(\nu)\omega](e_1, e_2),$$

where $e_1 = \partial/\partial \xi^1$ and $e_2 = \partial/\partial \xi^2$. Thus

$$\int_W \text{div} \, X \omega = \int_M \langle X, \nu \rangle j^* [i(\nu)\omega].$$

g) We choose $X = f \nabla f$ (on $\mathbb{R}^n$, the gradient $\nabla f$ of $f$ is the vector $\nabla f = \sum_{i=1}^{n} \partial_i f \partial/\partial x^i$), and get

$$\int_W \sum_{i=1}^{3} \partial_i (f \partial_i f) \omega = \int_W \sum_{i=1}^{3} (\partial_i f)^2 \omega + \int_W f \sum_{i=1}^{3} \partial_i f \omega.$$

Thus

$$\int_M f \partial_\nu f j^* [i(\nu)\omega] = \int_W \sum_{i=1}^{3} (\partial_i f)^2 \omega \geq 0$$

according to the result above and the hypothesis $\sum_{i=1}^{3} \partial_i f = 0$ in $W$.

If $f \partial_\nu f = 0$ on $M$, we find that $|\nabla f| = 0$ in $W$. Thus $f$ is constant.
b) The proof is by contradiction. Suppose $\Omega$ is connected, so that there exists an arc $\gamma$ starting from $\theta_1$ which goes to $\theta_2$ without cutting across $M$ ($M \cap \gamma = \emptyset$). Now as $|\nabla f|(P) \neq 0$ at each point $P$ of $M$, we have for instance $f(x) < f(P) = 0$ for $x \in \theta_1$ and $f(x) > f(P) = 0$ for $x \in \theta_2$. Thus $f|_{\gamma}$ has the value $f(P)$ somewhere, and $\gamma$ cuts across $M$.

Solution to Problem 2.44.

a) We have
\[
d\omega = 3(x^2 + y^2 + z^2)^{-\frac{3}{2}} dx \wedge dy \wedge dz
- 3(x^2 + y^2 + z^2)^{-\frac{3}{2}} (x^2 + y^2 + z^2) dx \wedge dy \wedge dz = 0,
\]
and $\omega$ is closed.

b) Let $(\theta, \varphi)$ be spherical coordinates,
\[
\begin{align*}
x &= \cos \varphi \cos \theta, \\
y &= \cos \varphi \sin \theta, \\
z &= \sin \varphi.
\end{align*}
\]
Thus
\[
\begin{align*}
dx &= -(\sin \varphi \cos \theta d\varphi + \cos \varphi \sin \theta d\theta), \\
dy &= \cos \varphi \cos \theta d\theta - \sin \varphi \sin \theta d\varphi, \\
dz &= \cos \varphi d\varphi,
\end{align*}
\]
\[
\Psi^* \omega = \left[ \cos^3 \varphi \cos^2 \theta + \cos^3 \varphi \sin^2 \theta \\
+ \sin^2 \varphi (\cos^2 \theta \cos \varphi + \cos \varphi \sin^2 \theta) \right] d\theta \wedge d\varphi \\
= \cos \varphi d\theta \wedge d\varphi.
\]

If $\overrightarrow{\nu}$ is the unit exterior normal vector at a point of $S_2$, the basis $(\overrightarrow{\nu}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi})$ is positive. Thus
\[
\int_{S_2} \Psi^* \omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \varphi d\varphi \int_{0}^{2\pi} d\theta = 4\pi.
\]

c) $\omega$ is not homologous to zero on $\mathbb{R}^3 - \{0\}$. Otherwise we would have $\omega = d\gamma$ and $\Psi^* \omega = d\Psi^* \gamma$. Thus according to Stokes' formula we would have $\int_{S_2} \Psi^* \omega = 0$.

d) $(\star \alpha)_{23...n} = \alpha_1$, $(\star \alpha)_{13...n} = -\alpha_2$, $(\star \alpha)_{12...k...n} = (-1)^{k-1} \alpha_k$ ($\hat{k}$ means that $k$ is omitted), since the metric is the Euclidean metric $\varepsilon_{ij} = \delta_i^j$ (the Kronecker symbol).
Let us compute \( d(*\alpha) \):

\[
d(*\alpha) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial}{\partial x^k} (*\alpha)_{12...k...n} dx^1 \wedge \cdots \wedge dx^n
\]

\[
= \sum_{k=1}^{n} \frac{\partial \alpha_k}{\partial x^k} dx^1 \wedge \cdots \wedge dx^n.
\]

Since \( \alpha_i = x^i [\sum_{j=1}^{n} (x^j)^2]^{-\frac{3}{2}} \), we have

\[
\frac{\partial \alpha_i}{\partial x^i} = \left[ \sum_{j=1}^{n} (x^j)^2 \right]^{-\frac{3}{2}} - n(x^i)^2 \left[ \sum_{j=1}^{n} (x^j)^2 \right]^{-1-\frac{3}{2}}.
\]

Thus we verify

\[
\sum_{k=1}^{n} \frac{\partial \alpha_k}{\partial x^k} = 0.
\]

e) Integrate \( \alpha_1 dx^2 \wedge \cdots \wedge dx^n \) on the half sphere where \( x^1 > 0 \), with \( x^2, x^3, \ldots, x^n \) as coordinates:

\[
A_1 = \int_{S_{n-1} \cap \{x^1 > 0\}} \alpha_1 dx^2 \wedge \cdots \wedge dx^n
\]

\[
= \int_{B_{n-1}} \sqrt{1 - \sum_{i=2}^{n} (x^i)^2} dx^2 \cdots dx^n,
\]

where \( B_{n-1} \) is the unit ball in \( \mathbb{R}^{n-1} \). We have \( dx^2 \wedge \cdots \wedge dx^n = dx^2 \cdots dx^n \), since \( (\nu, \partial/\partial x^2, \ldots, \partial/\partial x^n) \) is a positive basis if \( \nu \) is the unit exterior normal vector at a point of \( S_{n-1} \cap \{x^1 > 0\} \). Set \( \rho = \sum_{i=2}^{n} (x^i)^2 \); we find that

\[
A_1 = \sigma_{n-2} \int_0^1 \rho^{n-2} \sqrt{1 - \rho^2} d\rho,
\]

where \( \sigma_{n-2} \) is the volume of \( S_{n-2} \), the unit sphere in \( \mathbb{R}^{n-1} \). Thus \( A_1 = \frac{1}{2} \text{vol}(B_n) \). When we integrate \( dx^2 \wedge \cdots \wedge dx^n = -dx^2 \cdots dx^n \) on the half sphere \( x^1 < 0 \), we find the same value, since then \( x^1 = -\sqrt{1 - \rho^2} \). Thus

\[
\int_{S_{n-1}} \alpha_1 dx^2 \wedge \cdots \wedge dx^n = \text{vol}(B_n)
\]

and

\[
\int_{S_{n-1}} \Psi^*(\ast \alpha) = n \text{vol} B_n = \sigma_{n-1}.
\]

\( \ast \alpha \) cannot be homologous to zero, since otherwise we would have \( \int_{S_{n-1}} \Psi^*(\ast \alpha) = 0 \).
There are two alternative short proofs without computation. Let
\[ \gamma = \left[ \sum_{i=1}^{n} (x^i)^2 \right]^{n/2} \alpha; \]
then
\[ \int_{S_{n-1}(1)} \Psi^*(\ast \alpha) = \int_{S_{n-1}(1)} \Psi^*(\ast \gamma) = \int_{B_n(1)} d(\ast \gamma) = n \text{ vol } B_n = \sigma_{n-1}. \]

Or, if we consider the unit normal vector \( \nu = x^i \frac{\partial}{\partial x^i} \) at \( x \in S^1_{n-1}, \) then
\[ \ast \gamma = i(\nu)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \text{ and } \int_{S_{n-1}(1)} \Psi^*(\ast \gamma) = \text{area of } S_{n-1}(1) = \sigma_{n-1}. \]

**Solution to Exercise 2.45.**

a) We saw that the set \( T(n, n) \) of \( n \times n \) matrices \( M \) is in bijection with \( \mathbb{R}^{n^2}. \) If \( M = ((a_{ij})) \) and \( \{x^k\} \) is a coordinate system in \( \mathbb{R}^{n^2}, \) the bijection may be defined by \( x^k = a_{ij} \) with \( k = n(i - 1) + j. \) Then \( M \in O(n) \) if and only if
\[ B_j = \sum_{i=1}^{n} (a_{ij})^2 = 1 \text{ and } B_{jk} = \sum_{i=1}^{n} a_{ij} a_{ik} = 0 \]
for all \( j \) and all \( k > j. \)

There are \( \frac{n(n+1)}{2} \) equations. If we prove that the map
\[ \Gamma : M \rightarrow (B_1, \cdots, B_n, B_{12}, \cdots, B_{1n}, B_{23}, \cdots, B_{(n-1)n}) \]
is of rank \( \frac{n(n+1)}{2} \) on \( O(n), \) Theorem 1.19 will imply that \( O(n) \) is a submanifold of \( \mathbb{R}^{n^2} \) of dimension \( \frac{n(n-1)}{2}. \)

We have
\[ \frac{\partial B_j}{\partial a_{ij}} = 2a_{ij}, \quad \frac{\partial B_{jk}}{\partial a_{ij}} = a_{ik}, \quad \frac{\partial B_{jk}}{\partial a_{ik}} = a_{ij}. \]
At \( I \) we obtain
\[ \frac{\partial B_j}{\partial a_{jj}} = 2, \quad \frac{\partial B_{jk}}{\partial a_{jk}} = 1, \quad \frac{\partial B_{jk}}{\partial a_{kj}} = 1. \]
The other derivatives vanish.

If we suitably arrange the \( B_j \) and \( B_{jk}, \) there is in \( D \Gamma \) a diagonal matrix with 2 and 1 on the diagonal. Thus \( \Gamma \) is of rank \( \frac{n(n+1)}{2} \) at \( I. \)
Since \( \Gamma \) is \( C^\infty, \) \( \Gamma \) is of rank \( \frac{n(n+1)}{2} \) in a neighbourhood of \( I \) in \( O(n). \)
So this neighbourhood is a submanifold of \( \mathbb{R}^{n^2} \) of dimension \( \frac{n(n-1)}{2}. \)

Now consider the question in a neighbourhood \( V \) of \( A \in O(n). \)
The map of \( V \) into \( T(n, n) \) defined by \( M \rightarrow tAM \) is a homeomorphism of \( V \) onto a neighbourhood of \( I. \) Thus \( V \cap O(n) \) is a submanifold of \( T(n, n) \) of dimension \( \frac{n(n-1)}{2}. \) As \( A \) is arbitrary, \( O(n) \) is a submanifold of \( T(n, n) \) of dimension \( \frac{n(n-1)}{2}. \)
We can give an alternative proof. Let \( \varphi \) be the map of \( GL(\mathbb{R}^n) \) into the set \( \Sigma \) of symmetric matrices, defined by \( \varphi : M \rightarrow {}^tMM \). \( GL(\mathbb{R}^n) \) is an open set of \( T(n,n) \), thus a submanifold, and it is easy to see that \( \Sigma \) is also a submanifold of \( T(n,n) \). We have \( (D\varphi)_M(H) = {}^tMH + {}^tHM \). The rank of \( \varphi \) at \( M \in O(n) \) is equal to \( \frac{n(n+1)}{2} \), the dimension of \( \Sigma \), since \( (D\varphi)_M \) is surjective. Indeed, let \( B \in \Sigma \); then \( \frac{1}{2}MB \) satisfies \( (D\varphi)_M(\frac{1}{2}MB) = \frac{1}{2}MMB + \frac{1}{2}B^tMM = B \). Thus \( O(n) = \varphi^{-1}(I) \) is a submanifold of \( GL(\mathbb{R}^n) \) of dimension \( \frac{n(n-1)}{2} = n^2 - \frac{n(n+1)}{2} \), since \( \dim GL(\mathbb{R}^n) = n^2 \).

b) Let \( t \rightarrow M(t) \) be a differentiable curve in \( O(n) \) such that \( M(0) = I \). Set \( M(t) = ((a_{ij}(t))) \). \( a_{ij}(0) = \delta^i_j \), the Kronecker symbol. We have
\[
\left( \frac{dB_j(t)}{dt} \right)_{t=0} = 2a'_{jj}(0) \quad \text{and} \quad \left( \frac{dB_{jk}(t)}{dt} \right)_{t=0} = a'_{kj}(0) + a'_{jk}(0).
\]
Since \( B_j(t) \) and \( B_{jk}(t) \) are constants, it follows that \( a'_{jj}(0) = 0 \) and \( a'_{kj}(0) = a'_{jk}(0) = 0 \) for all \( j \) and \( k \neq j \). Thus \( dM(t)/dt \) is a tangent vector of \( O(n) \) at \( I \). This vector may be identified with the antisymmetric matrix \( ((a'_{ij}(0))) \). Thus \( T_I(O(n)) \) may be identified with the set \( S \) of antisymmetric \( n \times n \) matrices.

c) Let \( u \rightarrow M(u) \) be a differentiable curve in \( O(n) \) such that \( M(0) = M \). Then \( u \rightarrow \tilde{M}(u) = {}^tMM(u) \) is a differentiable curve in \( O(n) \) through \( I, \tilde{M}(u) = MM(u) \), and \( d\tilde{M}/du = M dM/du \). The result follows.

We saw just above that \( \Psi \) is bijective. Moreover \( \Psi \) is differentiable. \( \Psi^{-1} \) is the map of \( T(O(n)) \) onto \( O(n) \times S : (M, B) \rightarrow (M, {}^tMB) \), and this map is differentiable.

d) \( T(O(n)) \) is trivial since it is diffeomorphic to \( O(n) \times \mathbb{R}^{\frac{n(n-1)}{2}} \). Indeed, \( S \) may be identified to \( \mathbb{R}^{\frac{n(n-1)}{2}} \), since \( S \) is defined by \( \frac{n(n+1)}{2} \) linear equations in \( T(n,n) \) identified to \( \mathbb{R}^{n^2} \). Moreover, \( \Psi^{-1} \) is linear on each fiber. Thus \( T(O(n)) \) is parallelisable.

Solution to Problem 2.46.

a) We have \( dw = df \wedge dg = d \log |f| \wedge \omega = d\theta \wedge \omega \), with \( \theta = \log |f| \) in a neighbourhood where \( f \) does not vanish.

b) We have \( dw = dz \wedge (ydx + xdy) \). With \( f = e^{-xy} \),
\[
d \log f \wedge \omega = -(ydx + ydx) \wedge [(ydx + xdy)z + dz]
= dz \wedge (ydx + xdy).
\]
We have indeed \( dw = d \log e^{-xy} \wedge \omega \). So
\[
dg = e^{xy}(yzdx + xzdy + dz) = d(ze^{xy}),
\]
and we can choose $g = ze^{xy}$.

c) If $f$ and $g$ exist, we can write $d\omega = \theta \wedge \omega$. Now here $d\omega = dx \wedge dy$ does not have a term with $dz$ like $\theta \wedge \omega$ with $\theta$ not zero. So $f$ and $g$ do not exist.

d) Since $\omega(0) \neq 0$, there is a coefficient of the 1-form $\omega$ which does not vanish on a neighbourhood $V$ of $0 \in \mathbb{R}^{n+1}$: $\omega = B(x, z)dz + \cdots$ with $B(0, 0) \neq 0$. Set $\tilde{\omega} = \omega/B(x, z)$. If we can write $\tilde{\omega} = \tilde{f}dg$, then $\omega = fdz$ with $f = B(x, z)\tilde{f}$, and $\tilde{\omega}$ has a simpler expression.

e) According to Cauchy's theorem, $F(t, a, v)$ exists, and depends differentially on $t$ and on the parameter $(a, c)$.

Consider $b = \lambda a$ with $\lambda \in \mathbb{R}$, and let $F(u, b, v)$ be a solution of the equation

$$\frac{\partial F}{\partial t} = \sum_{i=1}^{n} A_i(a\lambda u, F)\lambda a^i, \quad F(0, b, c) = c.$$ 

Setting $t = \lambda u$ yields

$$\frac{\partial F}{\partial t} = \sum_{i=1}^{n} A_i(at, F)a^i, \quad F(0, a\lambda, c) = c.$$ 

Thus, since the solution of the equation is unique,

$$F(\lambda u, a, v) = F(u, \lambda a, v).$$

Pick $u = 1$, $\lambda = t$. This gives us the result.

f) Since $F(0, a, c) = c$ in $J$, $\frac{\partial F}{\partial u}(0, a, c) = 1$.

Thus $\frac{\partial F}{\partial u}(t, a, v) \neq 0$ in a neighbourhood of $(0, a, c)$. Applying the inverse function theorem, we can express $v$ in terms of $u$ and $z$. $(u, v)$ form a coordinate system in a neighborhood of $(0, c)$. If $dv = 0$ ($c$ is given) and $u = at$ with $a = \text{constant}$, then $\omega = 0$ according to the definition of $F$. Thus $P_i(at, v)a^i = 0$.

g) $\Psi^*\omega = \sum_{i=1}^{n} P_i(at, v)tda^i + B(at, v)dv$ according to the result above. Thus $R_i(t, a, v) = tP_i(at, v)$.

h) We have $d(\Psi^*\omega) = \Psi^*d\omega = \Psi^*\theta \wedge \Psi^*\omega$. Thus

$$\sum_{i=1}^{n} dR_i \wedge da^i + dB \wedge dv = \Psi^*\theta \wedge \left(\sum_{i=1}^{n} R_i da^i + Bdv\right).$$

Since the coefficient of $dt \wedge da^i$ must be the same on both sides,

$$\frac{\partial R_i}{\partial t} = HR_i$$

and

$$R_i(0, a, v) = 0.$$ 

The solution of this equation is unique according to Cauchy's theorem. Thus $R_i \equiv 0$ and $\omega = B(u, v)dv$. 

Solution to Exercise 2.47.

a) Let $u = \sum_{i=1}^{n} (u^i \frac{\partial}{\partial x^i} + v^i \frac{\partial}{\partial y^i})$. Then $\Pi_* u = \sum_{i=1}^{n} u^i \frac{\partial}{\partial x^i}$.

Let $\alpha = \sum_{i=1}^{n} \alpha_i dx^i$. Then $\langle \Pi_* u, \alpha \rangle = \sum_{i=1}^{n} \alpha_i u^i$.

Thus $\omega = \sum_{i=1}^{n} y^i d\xi^i$, since $y^i = \alpha_i$ at $\alpha$.

b) We have $d\omega = \sum_{i=1}^{n} dy^i \wedge d\xi^i$ and $\Omega = n! dy^1 \wedge d\xi^1 \wedge dy^2 \wedge \cdots \wedge d\xi^n$, which is nonzero everywhere. Hence $T^*(M)$ is orientable.
Chapter 3

Integration of Vector Fields and Differential Forms

The first part of this chapter concerns the integration of vector fields. As a vector field is, by definition, a section of $T(M)$, its components in a local chart are differentiable functions. So, by the Cauchy theorem, we prove that a vector field $X$ is integrable. It defines a one-parameter local group of local diffeomorphisms. That allows us to define $\mathcal{L}_X$, the Lie derivative with respect to the vector field $X$, of a vector field $Y$ or a differential form $\omega$.

Instead of considering a vector field $X(x)$ (1-direction field), we can consider $p$-direction fields $H_x$ ($1 < p < n$) and ask the question: Do there exist integral manifolds $W$ of dimension $p$? That is, do there exist submanifolds $W$ such that $T_x(W) = H_x$ for all $x \in W$?

Frobenius' theorem states that, if a certain necessary and sufficient condition is satisfied, there exists, through a given point $x_0$, a unique integral manifold of dimension $p$.

Integration of Vector Fields

3.1. Definition. A differentiable map $(t, P) \rightarrow \varphi(t, P) = \varphi_t(P)$ of $\mathbb{R} \times M$ into a differentiable manifold $M$ is called a one parameter group of diffeomorphisms on $M$ if

a) $\varphi(0, P) = P$, $\varphi_0 = identity$, and

b) for all $s$ and $t$ belonging to $\mathbb{R}$, $\varphi_{s+t} = \varphi_s \circ \varphi_t$. 

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φₜ is indeed a diffeomorphism, since φₜ ° φ₋ₜ = φ₋ₜ ° φₜ = φ₀.

A one parameter group of diffeomorphisms on M defines a vector field on M by \( X_P = [dφₜ(P)/dt]_{t=0} \). It is the tangent vector at \( t = 0 \) to the trajectory \( γ \) of \( P: t → γ(t) = φₜ(P) \).

The tangent vector at \( φ_s(P) \) to \( γ \) is

\[
\left[ \frac{dφ_t(P)}{dt} \right]_{t=s} = \left[ \frac{dφ_{s+u}(P)}{du} \right]_{u=0} = \left[ \frac{dφ_u[φ_s(P)]}{du} \right]_{u=0} = X_{φ_s(P)}.
\]

Moreover, since \( φ_{s+u}(P) = φ_s[φ_u(P)] \), we have

\[ X_{φ_s(P)} = dφ_s \left[ \frac{dφ_u(P)}{du} \right]_{u=0} = (φ_s)_*X_P. \]

3.2. Definition. A differentiable map \((t, P) → (t, P) = ϱ(t, P)\) of an open neighbourhood of \( 0 \times M \subset \mathbb{R} \times M \), into \( M \) is called a one parameter local group of local diffeomorphisms \( ϱ \) of \( M \) if

a) \( ϱ(0, P) = P, \) \( ϱ_0 \) = identity, and

b) for all \( P \in M \) and all \( t \) and \( s \) in \( \mathbb{R} \), \( ϱ(t+s, P) = ϱ[s, ϱ(t, P)] \) whenever \((t, P), (t + s, P) \) and \([s, ϱ(t, P)] \) are in \( Ω \).

Let us verify that \( ϱ \) is indeed a local diffeomorphism. Let \( t \neq 0 \) be a real number such that \( ϱ(ut, P) \) exists for all \( u \in [0,1] \). \( P \) is fixed. Consider a compact neighbourhood \( B \) of the curve \( C = \{ ϱ(ut, P) \mid u \in [0,1] \} \), and \( ε > 0 \) such that \( ] - ε, ε[ \times B \subset Ω \). We verify easily that \( B \) and \( ε \) exist, since \( C \) is a compact set.

Let \( q \in \mathbb{N} \) be such that \( |t| ≤ qε \). We can write

\[ ϱ(P) = (ϱ_q \circ ϱ_{q-1} \circ \cdots \circ ϱ_1)(P), \]

the product of \( q \) factors \( ϱ_q \), which is invertible. Thus \( ϱ_q \) is everywhere of rank \( n \) on \( C \). Consequently, \( ϱ \) is of rank \( n \) at \( P \), it is locally invertible at \( P \), and

\[ (ϱ_q)^{-1} = ϱ_{-q} \circ ϱ_{-q+1} \circ \cdots \circ ϱ_{-1} \] (q times).

A one parameter local group of local diffeomorphisms \( ϱ \) defines a vector field \( X_P = [dϱ_t(P)/dt]_{t=0} \). The converse is the goal of

3.3. Theorem. A \( C^{r-1} \) vector field \( X \) on \( M, r > 1 \), defines a one parameter local group of local diffeomorphisms obtained by integrating the following differential equation on \( M \):

\[ \frac{dφ_t(P)}{dt} = X_{φ_t(P)}. \]

An integral curve of \( X \) is a differentiable curve \( γ(t) \) such that

\[ \frac{dγ(t)}{dt} = X_{γ(t)}. \]
Let \((\theta, \Psi)\) be a local chart with coordinate system \(x^1, x^2, \ldots, x^n\), and let \(P \in \theta\). The equation is, denoting \(x^i[\varphi_t(P)]\) by \(x^i(t)\),
\[
\frac{dx^i}{dt} = X^i(x^1, x^2, \ldots, x^n).
\]

The Cauchy theorem asserts that there exist a neighbourhood \(I \times U\) of \((0, P)\) in \(\mathbb{R} \times \theta\) and a unique differentiable map \(\varphi(t, Q)\) of \(I \times U\) into \(\theta\) satisfying the equation above and \(\varphi(0, Q) = Q\) for all \(Q\) in \(U\). Moreover, since \(d\varphi_{s+t}/dt = d\varphi_{s+t}/ds\) = \(X\varphi_{s+t}\), uniqueness implies that \(\varphi_{s+t}(P) = \varphi_t[\varphi_s(P)]\).

Thus we exhibit a cover of \(M\) by open sets \(U_j, j \in J\), such that on \(I_j \times U_j\) there is a map \(\varphi_j\) with the properties defined above. Set \(\Omega = \bigcup_{j \in J}(I_j \times U_j)\) (\(\Omega\) is an open neighbourhood of \(0 \times M\) in \(\mathbb{R} \times M\)), and set \(\varphi(t, P) = \varphi_j(t, P)\) if \((t, P) \in I_j \times U_j\). This definition of \(\varphi\) makes sense because of the uniqueness of the solution.

### Lie Derivative

#### 3.4. Definition

Let \(X\) and \(Y\) be two \(C^2\) vector fields on a differentiable manifold \(M\), and \(\varphi_t\) the local group of local diffeomorphisms related to \(X\). The vector field
\[
\mathcal{L}_X(Y) = \lim_{t \to 0} \frac{1}{t}[(\varphi_{-t})_* Y_{\varphi_t(P)} - Y_P]
\]
is called the **Lie derivative** of the vector field \(Y\) with respect to the vector field \(X\). Likewise we define the **Lie derivative** of contravariant tensor fields.

The **Lie derivative**, with respect to the vector field \(X\), of a function \(f\) is
\[
\mathcal{L}_X(f) = \lim_{t \to 0} \frac{1}{t}(f \circ \varphi_t - f),
\]
and that of a differential form \(\omega\) is
\[
\mathcal{L}_X(\omega) = \lim_{t \to 0} \frac{1}{t}[(\varphi_t)^* \omega_{\varphi_t(P)} - \omega_P].
\]

Let us verify that \(\mathcal{L}_X(f) = X(f)\). \(\mathcal{L}_X(f)\) is the differential at \(t = 0\) of \(f \circ \varphi_t\). Now
\[
\left(\frac{d(f \circ \varphi_t)}{dt}\right)_{t=0} = df \circ \left(\frac{d\varphi_t}{dt}\right)_{t=0} = X(f).
\]

In a neighbourhood of \(P\) we saw that for \(t\) small enough \(\varphi_t\) and \(\varphi_{-t}\) exist. \((\varphi_{-t})_*\) at \(P\) is a map of \(T_{\varphi_t(P)}\) onto \(T_P(M)\). So in the definition of \(\mathcal{L}_X(Y)\) we have the difference of two vectors at \(P\) in the bracket. For differential forms we must use \((\varphi_t)^*\) to have the difference of two forms at \(P\) in the bracket.

#### 3.5. Proposition

\(\mathcal{L}_X(Y) = [X, Y]\).
3. Integration of Vector Fields and Differential Forms

Proof. Let \( f \) be a \( C^2 \) function in a neighbourhood \( \Omega \) of \( P \). Compute \([\mathcal{L}_X(Y)]_P(f)\). For \( t \) small, we can write

\[
[(\varphi_{-t})_*Y_{\varphi_{t}(P)}](f) - Y_P(f) = [Y_{\varphi_{t}(P)}](f \circ \varphi_{-t}) - [Y_{\varphi_{t}(P)}](f) + [Y_{\varphi_{t}(P)}](f) - Y_P(f).
\]

The result follows. Indeed,

\[
\lim_{t \to 0} \frac{1}{t} \{[Y(f)]_{\varphi_{t}(P)} - [Y(f)]_P\} = X[Y(f)]_P
\]

and

\[
\lim_{t \to 0} \frac{1}{t} \{[Y_{\varphi_{t}(P)}](f \circ \varphi_{-t}) - [Y_{\varphi_{t}(P)}](f)\} = -\lim_{t \to 0} \left[ Y \left( \frac{f \circ \varphi_{-t} - f}{-t} \right) \right]_{\varphi_{t}(P)} = -Y[X(t)]_P.
\]

\( t \) is small enough so that \( \varphi_{t}(P) \) and \( \varphi_{-t}(P) \) belong to \( \Omega \). \([Y_{\varphi_{t}(P)}](f)\) is the value of \( Y(f) \) at \( \varphi_{t}(P) \).

3.6. Proposition. a) If \( u \) and \( v \) are two contravariant tensor fields, then

\[
\mathcal{L}_X(u \otimes v) = \mathcal{L}_X(u) \otimes v + u \otimes \mathcal{L}_X(v).
\]

b) If \( \omega \) and \( \omega' \) are two differential forms, then

\[
\mathcal{L}_X(\omega \wedge \omega') = \mathcal{L}_X(\omega) \wedge \omega' + \omega \wedge \mathcal{L}_X(\omega').
\]

On differential forms,

\[
d\mathcal{L}_X = \mathcal{L}_X d \quad \text{and} \quad \mathcal{L}_X = i(X)d + di(X).
\]

The proof of (a) is similar to the proof of the first part of (b). Since \( \varphi^*(\omega \wedge \omega') = (\varphi^* \omega) \wedge (\varphi^* \omega') \), we can write

\[
(\varphi_t^*)(\omega \wedge \omega')_{\varphi_t(P)} - \omega_P \wedge \omega'_P
\]

\[
= (\varphi_t^* \omega_{\varphi_t(P)}) \wedge [\varphi_t^* \omega'_{\varphi_t(P)} - \omega'_P] + [\varphi_t^* \omega_{\varphi_t(P)} - \omega_P] \wedge \omega'_P.
\]

The first result in (b) follows. For the second we use Proposition 2.26: \( \varphi^*d = d\varphi^* \). Thus

\[
\mathcal{L}_X(d\omega) = \lim_{t \to 0} \frac{1}{t} \{[(\varphi_t)^*(d\omega)]_{\varphi_t(P)} - (d\omega)_P\}
\]

\[
= \lim_{t \to 0} \frac{1}{t} d[(\varphi_t)^*(\omega)]_{\varphi_t(P)} - \omega_P\]

\[
= d\mathcal{L}_X \omega.
\]

And now let us prove the last result. It is true for functions: \( i(X)f = 0 \)

and \( i(X)df = X(f) = \mathcal{L}_X(f) \).

For differential 1-forms, such as \( df \),

\[
d[i(X)df] = dX(f) = d\mathcal{L}_X(f) = \mathcal{L}_X(df) \quad \text{and} \quad [i(X)d]df = 0.
\]
Since $\mathcal{L}_X$ is linear, we have to prove the result only for $p$-forms such as $\omega = fdx^{i_1} \wedge \cdots \wedge dx^{i_p}$:

$$\mathcal{L}_X \omega = \mathcal{L}_X (fdx^{i_1} \wedge \cdots \wedge dx^{i_p})$$

$$+ f \sum_{j=1}^{p} dx^{i_1} \wedge \cdots \wedge dx^{i_j-1} \wedge \mathcal{L}_X dx^{i_j} \wedge dx^{i_{j+1}} \wedge \cdots \wedge dx^{i_p}.$$ 

Observe that $\mathcal{L}_X dx^i = d\mathcal{L}_X x^i = dX(x^i) = dX^i$, where $\{X^i\}$ are the components of the vector $X$ in the basis $\{\partial/\partial x^i\}$. Moreover,

$$i(X)\omega = f \sum_{j=1}^{p} (-1)^{j-1} dx^{i_1} \wedge \cdots \wedge dx^{i_j-1} \wedge X^j dx^{i_{j+1}} \wedge \cdots \wedge dx^{i_p},$$

$$di(X)\omega = df \wedge \sum_{j=1}^{p} (-1)^{j-1} dx^{i_1} \wedge \cdots \wedge dx^{i_j-1} \wedge X^j dx^{i_{j+1}} \wedge \cdots \wedge dx^{i_p}$$

$$+ f \sum_{j=1}^{p} dx^{i_1} \wedge \cdots \wedge dx^{i_j-1} \wedge dX^j \wedge dx^{i_{j+1}} \wedge \cdots \wedge dx^{i_p}$$

and

$$i(X)d\omega = X(f)dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$+ df \wedge \sum_{j=1}^{p} (-1)^j dx^{i_1} \wedge \cdots \wedge dx^{i_j-1} \wedge X^j dx^{i_{j+1}} \wedge \cdots \wedge dx^{i_p}.$$ 

Thus $\mathcal{L}_X \omega = [i(X)d + di(X)]\omega$.

The Frobenius Theorem

3.7. Definition. A $p$-direction field $H_x$ on $M$ is defined by prescribing, at each point $x$ of $M$, a vector subspace $H_x$ of dimension $p$ of $T_x(M)$ which satisfies a differentiability condition. We can write this condition in one of the following forms:

a) Each $x_0 \in M$ has a neighbourhood $V$ where there exist $p$ differentiable vector fields $X_1, X_2, \cdots, X_p$ such that $X_1(x), X_2(x), \cdots, X_p(x)$ form a basis of $H_x$ for all $x \in V$.

b) Each $x_0 \in M$ has a neighbourhood $V$ where there exist $q = n - p$ differential 1-forms $\omega_1, \omega_2, \cdots, \omega_q$ such that

$$X \in H_x \iff \omega_1(X) = \omega_2(X) = \cdots = \omega_q(X) = 0.$$ 

3.8. The Frobenius Theorem. In order that, for each point $x_0 \in M$, there exist a submanifold $W_{x_0} \ni x_0$ of dimension $p$ (called the integral manifold through $x_0$) of a neighbourhood $V$ of $x_0$, tangent to $H_x$ in each point $x \in W_{x_0}$, it is necessary and sufficient that $[X_i, X_j]_x \in H_x$ for all $i, j$ and
$x$ ($1 \leq i \leq j \leq p$) if we consider condition (a). On the other hand, if we consider condition (b), a necessary and sufficient condition for the existence of an integral manifold through any given point is that

$$d\omega_i \wedge \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_q = 0 \quad \text{for all } i \ (1 \leq i \leq q).$$

We will give below a proof of Frobenius' theorem and some developments of Pfaff systems.

If $p = 1$, we have to integrate vector fields. We saw that this is always possible. We can verify that the conditions of Frobenius' theorem are satisfied when $p = 1$. In this case $H_x$ has dimension 1. Let $X \neq 0$ be a vector field such that $X(x) \in H_x$; when $x \in V$, we have $[X, X]_x = 0 \in H_x$. The condition written with differential 1-forms is also trivially satisfied. As $q = n - 1$, $d\omega_i \wedge \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{n-1}$ is a differential $(n + 1)$-form which vanishes.

But when $p > 1$, intuitively there is no integral manifold in general (for example, when $p = 2$ and $n = 3$). Let $X, Y$ be two vector fields generating $H_x$ for $x \in V$.

From $x_0$, we integrate $X$, then $Y$. We get $C_0$ and $\tilde{C}_0$. Let $C_1$ be the integral curve of $X$ from a point $x_1$ of $\tilde{C}_0$, and let $C_2$ be the integral curve of $Y$ from a point $x_2$ of $C_0$. There is no reason that $C_2$ should intersect $C_1$. But if there is an integral surface $S$ through $x_0$, then $C_1$ meets $C_2$ since they are included in $S$.

**3.9. Definition.** A system of exterior differential equations on $M$ is a set of $N$ equations \{$\omega^\alpha = 0$\}, $\alpha = 1, 2, \cdots, N$, where $\omega^\alpha$ are exterior differential $p_\alpha$-forms. The solutions of the system are vector subspaces $T$ such that $\omega^\alpha(X_1, X_2, \cdots, X_{p_\alpha}) = 0$ for all $\alpha$ whenever $X_1, X_2, \cdots, X_{p_\alpha}$ belong to $T$. 
The pair \((W_p, \Phi)\) consisting of a manifold and of a differentiable map \(\Phi\) of \(W_p\) in \(M_n\) is an integral manifold of the system if \(\Phi^*\omega^\alpha = 0\) for all \(\alpha\).

Consider the ideal \(I\) of the exterior algebra \(\Lambda\) generated by the \(\omega^\alpha:\)

\[ \omega \in I \iff \omega = \sum_{1 \leq \alpha \leq N} \omega^\alpha \wedge \theta^\alpha \text{ with } \theta^\alpha \in \Lambda. \]

**3.10. Proposition.** Every solution of the system \(\{\omega^\alpha = 0\}\) is a zero of each form of \(I\), and conversely.

**3.11. Definition.** Two systems of exterior differential equations are said to be algebraically equivalent, \(\{\omega^\alpha = 0\} \iff \{\overline{\omega}^\beta = 0\}\), if they generate the same ideal. Then each element \(\overline{\omega}^\beta = \sum_\alpha \omega^\alpha \wedge \theta^\alpha\), and vice versa.

According to Proposition 3.10, the two systems have the same solutions.

**3.12. Definition.** The differential system \(\{d\omega^\alpha = 0, \omega^\alpha = 0\}\) is called the closure of the system \(\{\omega^\alpha = 0\}\).

**3.13. Proposition.** A system of exterior differential equations and its closure have the same integral manifolds.

Indeed, \(\Phi^*\omega^\alpha \equiv 0\) implies \(\Phi^*d\omega^\alpha = d(\Phi^*\omega^\alpha) \equiv 0\), since \(d\) and \(\Phi^*\) commute.

If two systems are algebraically equivalent, so are their closures. Indeed, if \(\overline{\omega}^\beta = \sum_\alpha \omega^\alpha \wedge \theta^\alpha\), then \(d\overline{\omega}^\beta = \sum_\alpha d\omega^\alpha \wedge \theta^\alpha + \sum_\alpha (-1)^p \omega^\alpha \wedge d\theta^\alpha\).

**3.14. Definition.** A system is said to be closed if it is algebraically equivalent to its closure: \(dI \subset I\). The closure of a system is closed (\(d^2 = 0\)).

**3.15. Definition.** A system of exterior differential equations is called a Pfaff system if the \(\omega^\alpha\) are all differential 1-forms. The equations \(\{\omega^\alpha = 0\}\) determine at one point a vector subspace which contains the tangent space of all integral submanifolds \(W_p \subset M_n\). A Pfaff system of rank \(q\), in an open set \(\Omega\), is said to be completely integrable if there exist \(q\) differentiable functions \(y^\beta (\beta = 1, 2, \ldots, q)\) such that the system \(\{\omega^\alpha = 0\}\) \((\alpha = 1, 2, \ldots, N)\) is algebraically equivalent to the systems \(\{y^\beta = 0\}\) \((\beta = 1, 2, \ldots, q)\). That is to say, \(\omega^\alpha = a^\alpha_{\beta} dy^\beta\). The \(y^\beta\) are called first integrals of the Pfaff system.

In writing \(\omega^\alpha\) we omit the sign \(\sum\). This is the Einstein convention.

**The Einstein Convention.** When the same index (say \(\beta\)) is above and below, summation over this index is assumed (here from \(1\) to \(q\)). \(\beta\) is called a dummy index. Henceforth we will use the Einstein convention to simplify the writing.

**3.16. Theorem.** If a Pfaff system of rank \(q\) in \(\Omega \subset M_n\) is completely integrable in \(\Omega\), then through each point \(x_0 \in \Omega\) there is exactly one integral submanifold of dimension \(p = n - q\).
Proof. The equations \( y^\beta(x) = y^\beta(x_0) \) (1 \( \leq \beta \leq q \)) define a differentiable submanifold \( W \) of dimension \( p = n - q \) through \( x_0 \). Indeed, the system \( \{ \omega^\alpha = 0 \} \) being of rank \( q \), the system \( \{ dy^\beta = 0 \} \) is of rank \( q \), there are \( q \) functions \( y^\beta \). We verify easily that \( (W, i) \), \( i \) being the inclusion, is an integral manifold of the system, the unique integral manifold of dimension \( p \) through \( x_0 \).

3.17. The Frobenius Theorem (second version). A necessary and sufficient condition for a Pfaff system \( \{ \omega^\alpha = 0 \} \), of rank \( q \) in \( \Omega \), to be completely integrable in \( \Omega \), is that the system is closed in \( \Omega \).

Proof. Necessity. Let \( \{ y^\beta \} \) be the \( q \) first integrals, \( \omega^\alpha \equiv a_\beta^\gamma dy^\beta \), where \( A = ((a_\beta^\gamma)) \) is an invertible matrix. Let \( B((b_\alpha^\gamma)) \) be its inverse. Then

\[
  d\omega^\alpha \equiv da_\beta^\gamma \wedge dy^\beta \equiv b_\gamma^\rho da_\gamma^\rho \wedge \omega^\gamma.
\]

Thus, if \( I \) is the ideal generated by the \( \omega^\alpha \), then \( dI \subset I \).

Sufficiency. The proof proceeds by induction on the dimension \( n \). First, in a space of dimension \( q \), a system of rank \( q \) is completely integrable. Indeed, since \( \omega^j \equiv a_i^j dx^i \) with \( i, j = 1, 2, \ldots, q \), the system \( \{ \omega^j = 0 \} \) is equivalent to the system \( \{ dx^i = 0 \} \), the matrix \( A = ((A_i^j)) \) being invertible by hypothesis.

In the general case, when the dimension is greater than \( q \), suppose we have proved the theorem up to dimension \( n - 1 \geq q \). More precisely, if a Pfaff system \( \{ \tilde{\omega}^\alpha = 0 \} \), of rank \( q \) in \( \Omega \), is closed, the \( \tilde{\omega}^\alpha \) depending differentiably on \( n - 1 \) variables \( x^1, x^2, \ldots, x^{n-1} \) and on some parameters \( x^n, x^{n+1}, \ldots \), then there exist \( q \) first integrals \( z^\alpha \) which depend differentiably on the \( n - 1 \) variables and on the parameters.

Consider the Pfaff system of rank \( q \),

\[
  \{ \omega^\alpha \equiv \sum_{i=1}^n a_i^\alpha dx^i = 0 \} \quad (\alpha = 1, 2, \ldots, q),
\]

where the \( a_i^\alpha \) depend differentiably on \( n \) variables and some parameters.

Without loss of generality we can suppose that \( \{ \tilde{\omega}^\alpha = \sum_{i=1}^{n-1} a_i^\alpha dx^i = 0 \} \) is of rank \( q \). The forms \( \tilde{\omega}^\alpha \) are differential 1-forms on \( \mathbb{R}^{n-1} \) depending on \( x^n \) (at this stage, \( x^n \) is a parameter), and on some other parameters. \( \tilde{d} \) designates differentiation in \( \mathbb{R}^{n-1} \) and \( d \) differentiation in \( \mathbb{R}^n \):

\[
  d\omega^\alpha \equiv \tilde{d}\omega^\alpha + \theta^\alpha \wedge dx^n, \quad \theta^\alpha \in \Lambda^1(\Omega).
\]

Since by hypothesis \( d\omega^\alpha \in I \), the ideal generated by the \( \omega^\alpha \) in \( \Lambda(\Omega) \), it follows that \( \tilde{d}\omega^\alpha \in \tilde{I} \) the ideal generated by the \( \tilde{\omega}^\alpha \) in \( \Lambda(\Omega \cap \pi) \) (\( \pi \) being the hyperplane \( x^n = \text{Const} \)). Indeed, if \( d\omega^\alpha \equiv \sum_{\beta=1}^q \gamma_\beta^\alpha \wedge \omega^\beta \), we can write \( d\omega^\alpha \).
Integrability Criteria

in the form

\[ d\omega^\alpha = \sum_{\beta=1}^{q} \tilde{\gamma}_\beta^\alpha \wedge \bar{\omega}^\beta + \theta^\alpha \wedge dx^n \]

with \( \tilde{\gamma}_\beta^\alpha \in \Lambda^1(\Omega \cap \pi) \) and \( \ddot{\omega} = \sum_{\beta=1}^{q} \tilde{\gamma}_\beta^\alpha \wedge \bar{\omega}^\beta \).

Thus there exist \( q \) functions \( z^\alpha \), depending differentiably on the variables and the parameters, such that \( \bar{\omega}^\alpha = A^\alpha_\beta dz^\beta \), \( ((A^\alpha_\beta)) \) being an invertible matrix \( ((B^\beta_\alpha)) \) its inverse) whose elements \( A^\alpha_\beta \) depend differentiably on the variables and on the parameters. The system \( \{ \omega^\alpha = 0 \} \) is equivalent to the system \( \{ \bar{\omega}^\beta \equiv B^\beta_\alpha \omega^\alpha = 0 \} \), where \( \bar{\omega}^\beta \) is nothing else but \( \bar{\omega}^\beta \equiv d\bar{\omega}^\beta + B^\beta_\alpha a^\alpha_n dx^n \equiv dx^\beta + b^\beta dx^n \). Since this system is closed \( (d\bar{\omega}^\beta \in I) \),

\[ db^\beta \wedge dx^n = \theta^\alpha_\beta \wedge (dz^\alpha + b^\alpha dx^n) \]

with \( \theta^\alpha_\beta \) a differential 1-form which is a linear combination of the \( dx^\alpha \) and \( dx^n \) only. Indeed, in the coordinate system \( z^1, z^2, \ldots, z^q, x^{q+1}, \ldots, x^n \), \( \theta^\alpha_\beta \) cannot have a term in \( dx^i \) with \( q < i < n \), as otherwise we would have in the right hand side a term in \( d\bar{\omega}^\beta \wedge dx^\alpha \), but no such term in the left hand side. Consequently there is no term in \( dx^i \wedge dx^n \) in the right hand side, and \( b^\beta \) is a function of the \( z^\alpha \), of \( x^n \) and of the parameters only, \( \partial b^\beta / \partial x^i = 0 \) for \( q < i < n \). Thus the system \( \{ \bar{\omega}^\beta = 0 \} \) is an ordinary system of differential equations

\[ \frac{dz^\beta}{dx^n} = -\theta^\beta(z^1, z^2, \ldots, z^q, x^n) \quad (1 \leq \beta \leq q) , \]

which locally has a solution according to Cauchy’s theorem. Consider \( q \) independent first integrals \( y^\alpha(z^1, z^2, \ldots, z^q, x^n) \) \( (1 \leq \alpha \leq q) \) in a neighbourhood of the considered point. The \( y^\alpha \) depend differentiably on the parameters. The system \( \{ \omega^\alpha = 0 \} \) is equivalent to the system \( \{ dy^\alpha = 0 \} \) — it is completely integrable.

Integrability Criteria.

3.18. If \( \omega^1, \omega^2, \ldots, \omega^q \) are \( q \) independent Pfaff forms, a necessary and sufficient condition for the \( p \)-form \( \theta \) to belong to the ideal \( I \) generated by the \( \omega^\alpha \) \( (1 \leq \alpha \leq q) \) is that \( \theta \wedge \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^q = 0 \).

Obviously the condition is necessary; if \( \theta = \sum_{\alpha=1}^{q} \theta^\alpha \wedge \omega^\alpha \), \( \theta \) satisfies this condition. To prove the sufficiency, consider a basis of \( \Lambda^1(\Omega) \) with \( \omega^1, \omega^2, \ldots, \omega^q \) completed with \( n - q \) 1-forms \( \omega^{q+1}, \ldots, \omega^n \):

\[ \theta = \sum_{i_1 < i_2 < \cdots < i_p} a_{i_1, i_2, \ldots, i_p} \omega^{i_1} \wedge \omega^{i_2} \wedge \cdots \wedge \omega^{i_p} . \]
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Then \( \theta \wedge \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^q = 0 \) implies that \( a_{i_1, i_2, \ldots, i_p} = 0 \) whenever \( i_1 > q \). Thus \( \theta \in I \).

The necessary and sufficient condition of the Frobenius theorem is

\[
d\omega^\alpha \wedge \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^q = 0 \quad \text{for } 1 \leq \alpha \leq q.
\]

3.19. Remark. We saw already that in a space of dimension \( n \), a Pfaff system of rank \( n \) is always integrable. The criterion above shows that this is also the case if the rank of the system is equal to \( n - 1 \) (an \((n+1)\)-form is identically zero). We can prove this result directly. If the \( \omega^\alpha = \sum_{i=1}^{n} a^\alpha_i dx^i \) \((\alpha = 1, 2, \ldots, n - 1)\) are independent, we can suppose that the matrix \( ((a^\beta_j)) \) \((\beta = 1, 2, \ldots, n - 1)\) is invertible. The system \( \{\sum_{i=1}^{n} a^\alpha_i dx^i = 0\} \) is then equivalent to a system of the type \( \{dx^i/dx^n = A^i(x^j, x^n)\} \) with \( i, j = 1, 2, \ldots, n - 1 \). The Cauchy theorem gives us the first integrals.

3.20. Example. The Pfaff equation in \( \mathbb{R}^3 \),

\[
\omega = a_1 dx^1 + a_2 dx^2 + a_3 dx^3 = 0
\]
is completely integrable in a neighbourhood \( \Omega \) of a point \( x_0 \) if \( a_1, a_2 \) and \( a_3 \) do not vanish simultaneously and if \( \omega \wedge d\omega = 0 \)—that is, if rank \( \omega = 1 \) and if

\[
a_1 \left( \frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3} \right) + a_2 \left( \frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1} \right) + a_3 \left( \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} \right) = 0.
\]

This can be written as \( \vec{A} \cdot \text{rot} \vec{A} = 0 \) with \( \vec{A} \) the vector of components \( a_i \). Under this condition, in a neighbourhood of \( x_0 \), \( \omega = 0 \) is equivalent to \( d\varphi = 0 \) for some function \( \varphi \), and through \( x_0 \) there is one and only one integral manifold of dimension 2. Its equation is \( \varphi(x) = \varphi(x_0) \).

3.21. Example. The Pfaff equation in \( \mathbb{R}^2 \),

\[
\tilde{\omega} = x^1 dx^2 - x^2 dx^1 = 0,
\]
is completely integrable in \( \mathbb{R}^2 - \{0\} \). In \( \mathbb{R}^2 - \{0\} \) the rank of \( \tilde{\omega} \) is 1 (it is zero at 0). The integral manifolds are straight lines \( x^2 = kx^1 \) \((k \in \mathbb{R})\). Through each point \( x_0 \neq 0 \) there is one and only one integral manifold.

3.22. Proposition. If a \( p \)-direction field on \( M \) is defined by \( p \) vector fields \( X_i(x) \), \( i = 1, 2, \ldots, p \) (see Definition 3.7), the necessary and sufficient condition of the Frobenius theorem is \( [X_i, X_j]_x \in H_x \) for all \( i, j \), and \( x \) \((1 \leq i \leq j \leq p)\).

Indeed, consider \( q \) differential 1-forms \( \omega_\alpha \) \((\alpha = 1, \ldots, q = n - p)\) such that \( \omega_\alpha(X) = 0 \) for all \( \alpha \) is equivalent to \( X \in H_x \). The necessary and sufficient condition for the Pfaff system to be completely integrable is that
\[ dw^a \in I, \text{ the ideal generated by the } \omega^a, \text{ and, according to the definition of the differential } d, \]
\[ dw(X_i, X_j) = X_i[\omega(X_j)] - X_j[\omega(X_i)] - \omega([X_i, X_j]). \]
If \( dw \in I \), then \( dw(X_i, X_j) = 0 \) and we have \( \omega([X_i, X_j]) = 0 \); thus \( [X_i, X_j] \in H_x \).

If \( [X_i, X_j] \in H_x \), we have \( dw(X_i, X_j) = 0 \), which implies \( dw \in I \). To see this we consider a basis of \( \Lambda^1(\Omega) \) with \( \omega^1, \omega^2, \ldots, \omega^q \) completed with \( n - q = p \) differential 1-forms \( \omega^{q+1}, \ldots, \omega^{q+p} \) such that \( \omega^{q+i}(X_j) = \delta^j_i \) (the Kronecker symbol). If \( dw = \sum_{1 \leq k, l \leq n} a_{kl} \omega^k \wedge \omega^l \), then \( dw(X_i, X_j) = a_{q+i,q+j} \) if \( i < j \). We must have \( a_{ij} = 0 \) whenever \( i > q \). That is, \( dw \in I \).

Exercises and Problems

3.23. Problem. Let \( M \) and \( W \) be two \( C^\infty \) differentiable manifolds of the same dimension, and let \( (u, x) \rightarrow G_u(x) \) be a \( C^\infty \) differentiable map of \( ]a, b[ \times M \rightarrow W \) \( (]a, b[ \subset \mathbb{R}) \) such that, for fixed \( u \), \( G_u(x) \) is a diffeomorphism of \( M \) onto \( G_u(M) \subset W \). We denote by \( X_u(y) \) the tangent vector at \( y = G_u(x) \) to the differentiable curve \( ]a, b[ \ni t \rightarrow G_t(x) \).

a) Let \( \Omega \) be a differential 2-form on \( W \) \( (\Omega \in \Lambda^2(W)) \). Let \( \{x^i\} \) be a coordinate system in a neighbourhood \( U \) of \( x_0 \in M \) and \( \{y^\alpha\} \) a coordinate system in a neighbourhood \( \theta \) of \( y_0 = G_u(x_0) \). Express on \( U \) the components of \( G^*_u\Omega \) in terms of the components \( \Omega_{\alpha\beta} \) of \( \Omega \) on \( \theta \).

b) Compute \( G^*_u\mathcal{L}_{X_u}\Omega \).

c) Prove that \( \frac{\partial}{\partial u}(G^*_u\Omega) = G^*_u\mathcal{L}_{X_u}\Omega \).

d) From now on \( M = W = B_0(1) = B \), the ball of center 0 and radius 1 in \( \mathbb{R}^n \). We consider for \( u \in ]0, 1[ \) the homothety \( G_u \) \( (G_u(x) = ux) \).

On which open set is \( X_u(y) \) a vector field? What are its components?

e) Henceforth \( \Omega \) is closed \( (d\Omega = 0) \). For \( a, b \in ]0, 1[ \), prove that \( G^*_a\Omega - G^*_b\Omega \) is the differential of a differential 1-form \( \gamma(a, b) \).

f) What is the limit of \( G^*_a\Omega \) when \( a \rightarrow 0 \)?

g) Show that there exists \( \gamma \in \Lambda^1(B) \) such that \( \Omega = d\gamma \).

3.24. Problem. Let \( M \) be a \( C^\infty \) differentiable manifold of dimension \( n = 2m \), and let \( \Omega \in \Lambda^2(M) \). We assume that the rank of \( \Omega \) is \( 2m \), that is to say, that, at any point \( x \in M \), \( \omega = \Omega \wedge \Omega \wedge \cdots \Omega \neq 0 \), and that \( d\Omega = 0 \).

The differential forms are assumed to be \( C^\infty \).

a) Show that for any \( X \neq 0 \), \( i(X)\Omega \) is nonzero.
3. Integration of Vector Fields and Differential Forms

b) Exhibit a basis $e_i (i = 1, 2, \ldots, 2m)$ of $T_{x_0}M$, $x_0$ fixed, such that
\[
\Omega(e_{2k-1}, e_{2k}) = 1 \text{ for } k = 1, 2, \ldots, m \text{ and } \Omega(e_i, e_j) = 0 \text{ for the other pairs } e_i, e_j \text{ with } i < j. \]
What is the expression of $\Omega(x_0)$ in a coordinate system $\{x^i\}$ associated to the local chart $(V, \Psi)$ at $x_0$, if $(\partial/\partial x^i)_{x_0} = e_i$?

We suppose that $\Psi(V)$ is a ball of $\mathbb{R}^n$.

c) On $V$, consider the constant differential 2-form $\tilde{\Omega}$ such that $\tilde{\Omega} = \Omega$ at $x_0$. For $t \in [0, 1]$, set $\Omega_t = \Omega + t(\tilde{\Omega} - \Omega)$. Verify that there is a neighbourhood $W \subset V$ of $x_0$ on which $\Omega_t$ is of rank $2m$ for all $t \in [0, 1]$, and that there exists a differential 1-form $\gamma$ on $V$ such that $\gamma(x_0) = 0$ and $d\gamma = \Omega - \tilde{\Omega}$ (use Problem 3.23).

d) Show that the map $X \mapsto i(X)\Omega$ is an isomorphism of $\Gamma(M)$, the space of $C^\infty$ differentiable vector fields, on $\Lambda^1(M)$. Denote by $X_t$ the vector field defined by $i(X_t)\Omega_t = \gamma$. We admit that the map $\phi : [0, 1] \times M \to T(M)$ is $C^\infty$.

e) Prove that there is, on a neighbourhood $V_o$ of $(0, x_0)$ in $[0, 1] \times M$, a $C^\infty$ differentiable map $f_t(x)$ such that $\partial f_t(x)/\partial t = X_t(x)$, $f_0(x) = 1d$. Prove that $\frac{\partial}{\partial t}(f_t^*\Omega_t) = 0$ (use the result of Problem 3.23).

f) From the previously posed questions, deduce the existence of a coordinate system $\{y^i\}$ on a neighbourhood of $x_0$ such that
\[
\Omega = dy^1 \wedge dy^2 + dy^3 \wedge dy^4 + \cdots + dy^{2m-1} \wedge dy^{2m}.
\]
We will admit that there is $W$ such that $[0, 1] \times W \subset V_o$.

3.25. Problem. Let $(x, y, z)$ be a coordinate system on $\mathbb{R}^3$ where we consider the differential 1-form
\[
\omega = (1 - yz)dx - x(z + x)dy + (1 + xy)dz
\]
a) Is the equation $\omega = 0$ completely integrable? What is its rank? Is the set $M$ of points where $\omega \equiv 0$ a submanifold of $\mathbb{R}^3$?

b) Find a nonzero vector field $Y$ such that $\omega(Y) = 0$, $Y$ being of the form $Y = a\frac{\partial}{\partial x} + c\frac{\partial}{\partial z}$ with $a$ and $c$ polynomials in $x, y, z$ (its second component vanishes). Integrate the vector field $Y$. What are the integral curves?

c) Exhibit two independent first integrals $y$ and $u$ of the equations considered in (b).

Write $\omega$ in the coordinate system $(x, y, u)$. Was the result obtained foreseeable?

d) What are the integral manifolds of the equation $\omega = 0$? Verify that they are indeed submanifolds of $\mathbb{R}^3$. How many integral manifolds are there through a given point of $\mathbb{R}^3$?
e) Beginning by integrating a vector field $X$ whose first component is zero, prove again the previous result.

3.26. Problem. Let $E$ be a vector space of dimension $n$. Set $F = E \times \mathbb{R}$. The pair consisting of a point $(x, z) \in E \times \mathbb{R}$ and of a hyperplane $H$ through it is called a contact element of $F$. If the hyperplane is not parallel to the $Oz$ axis, we say that the contact element is regular.

a) Show that the set of regular contact elements may be identified to $F \times E^*$.

b) For a submanifold $W$ of $F$ of dimension $p$ ($p \leq n$), we say that the contact element $(x, z, H)$ is tangent to $W$ if $(x, z) \in W$ and $T_{(x,z)}(W) \subset H$.

Prove that the set of regular contact elements tangent to $W$ forms a submanifold $\tilde{W}$ of $F \times E^*$ of dimension $n$.

*Hint.* Consider $q$ equations $f_i(x, z) = 0$ which define $W$, with $f_i \in C^1$ such that $df_i$ (1 $\leq$ $i$ $\leq$ $q$) are independent when $(x, z) \in W$.

c) We denote by $\{x^j\}$ (1 $\leq$ $j$ $\leq$ $n$) the coordinates of $E$ and by $\{a_j\}$ (1 $\leq$ $j$ $\leq$ $n$) those of $E^*$. On $F \times E^*$ we consider the differential form $\omega = dz - a_j dx^j$.

Prove that $\tilde{W}$ is an integral manifold of the Pfaff equation $\omega = 0$.

d) Is it possible that the equation $\omega = 0$ has integral manifolds of dimension $d > n$?

3.27. Problem. On $\mathbb{R}^4$, endowed with the coordinate system $(x, y, z, t)$, we consider two vector fields,

$$X = t \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} - x \frac{\partial}{\partial t}$$

and

$$Y = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - t \frac{\partial}{\partial z} + z \frac{\partial}{\partial t}.$$  

a) On which submanifold $M$ of $\mathbb{R}^4$ do these two vector fields define a two-plane field? Is this two-plane field integrable on $M$?

b) We consider on $\mathbb{R}^4$ a third vector field

$$Z = -z \frac{\partial}{\partial x} + t \frac{\partial}{\partial y} + x \frac{\partial}{\partial z} - y \frac{\partial}{\partial t}.$$  

Integrate this vector field. Do the integral paths have any particular property?

c) Show that $X, Y, Z$ define at each point $P \in M$ a 3-hyperplane field $H_P$. Is this field $H_P$ integrable?

d) Find a differential 1-form $\omega$ such that for all $P \in M$ and $\xi \in T_P(M)$, $\omega(\xi) = 0 \iff \xi \in H_P$. 

e) If they exist, what are the integral manifolds of the questions asked in (a) and (c).

3.28. Problem. Consider in \( \mathbb{R}^4 \) the Pfaff system \( S \):

\[
\begin{aligned}
\omega_1 &\equiv [x^3 - (x^1)^2]dx^1 + x^1 x^2 dx^2 + dx^3 = 0, \\
\omega_2 &\equiv f dx^1 + g dx^2 = 0,
\end{aligned}
\]

where \( f \) and \( g \) are functions of the four coordinates \( x^1, x^2, x^3, x^4 \).

a) Find functions \( f \) and \( g \) for which the system \( S \) is completely integrable. Show that it is then equivalent to a system of rank 2 in \( \mathbb{R}^3 \).

b) Find the integral manifolds of dimension 2 of the system \( S \) when \( f = g = 1 \).

c) Consider the differential equation in \( \mathbb{R}^3 \)

\[
\omega \equiv [x^3 - (x^1)^2 - x^1 x^2]dx^1 \wedge dx^2 - dx^2 \wedge dx^3 + dx^3 \wedge dx^1 = 0.
\]

Write the integral manifold through the line

\[
\begin{aligned}
x^1 &= a^1 v, \\
x^2 &= a^2 v, \\
x^3 &= a^3 v.
\end{aligned}
\]

3.29. Problem. Let \( S = \{\omega^\alpha\}_{\alpha \in \Lambda} \) be a family of exterior forms on the vector space \( \mathbb{R}^n \) each of degree \( q_\alpha > 0 \), and \( I \) the ideal generated by these forms in \( \Lambda(\mathbb{R}^n) \). Denote by \( Q \) the smallest subspace of \( \Lambda^1(\mathbb{R}^n) \) such that \( S \subset \Lambda(Q) \). \( Q \) is associated to \( S \).

a) Prove the existence and uniqueness of \( Q \). Its dimension is called the rank of \( S \).

b) Establish that the 1-forms of \( S \) belong to \( Q \).

c) Show that a necessary and sufficient condition for an exterior p-form \( \omega \) to be a monomial (that is to say, \( \omega \) is the exterior product of linear forms) is that the rank of \( \tilde{S} = \{\omega\} \) is \( p \).

Verify that the rank of a nonvanishing 2-form on \( \mathbb{R}^3 \) is always 2.

d) Denote by \( H \) the subspace of \( \mathbb{R}^n \) such that \( i_X \theta = \theta(X) = 0 \) for all \( \theta \in Q \) when \( X \in H \). Show that \( i_X \omega^\alpha = 0 \) for all \( \alpha \in A \).

Hint. Consider a basis of \( \Lambda^1(\mathbb{R}^n) \) formed by a basis of \( Q \) completed with a basis dual to a basis of \( H \).

e) Prove that \( X \in H \) if and only if \( i_X \omega \in I \) for all \( \omega \in I \).

Hint. For sufficiency, proceed by induction on the rank of the forms.

f) Let \( S = \{\omega^1, \omega^2\} \) with \( \omega^1 = adx + bdy + cdz \) and \( \omega^2 = Cdx \wedge dy + Bdz \wedge dx + Ady \wedge dz \) on \( \mathbb{R}^3 \). According to question (e), write the three conditions for the vector \( X \) of components \( (x, y, z) \) to belong to \( H \). Here \( a, b, c, A, B \) and \( C \) are \( C^\infty \) functions on \( \mathbb{R}^3 \).
g) Find three 1-forms \( \{ \gamma^j \} (j = 1, 2, 3) \) such that \( \{ \gamma^j(X) = 0 \ (j = 1, 2, 3) \} \) is equivalent to \( X \in H \). Find sufficient conditions for \( S = \{ \gamma^1, \gamma^2, \gamma^3 \} \) to be of rank 2.

Henceforth \( S = \{ \omega^\alpha = 0 \}_{\alpha \in A} \) is a system of exterior differential forms on an open set \( \Omega \) of a differentiable manifold \( M_n \). Each point \( x \in \Omega \) is associated to a subspace \( Q_x \) of \( T^*_x(M) \) corresponding to \( S = \{ \omega^\alpha(x) \}_{\alpha \in A} \). We assume that there exist \( p \) differential 1-forms \( \gamma^i \) on \( \Omega \) forming at each point \( x \) a basis of \( Q_x \). \( P = \{ \gamma^i = 0 \}_{1 \leq i \leq p} \) is said to be associated to \( \Sigma \).

h) Show that an integral manifold of the Pfaff system \( P = \{ \gamma^i = 0 \}_{1 \leq i \leq p} \) is an integral manifold for the system \( \Sigma \).

i) Consider on \( \mathbb{R}^3 \) two systems \( \Sigma_1 = \{ \omega^1 = 0, \omega^2 = 0 \} \) and \( \Sigma_2 = \{ \omega^2 = 0 \} \) with \( \omega^1 = dx + 2dy \) and \( \omega^2 = dx \wedge dy + 2dx \wedge dz + 4dy \wedge dz \). What are the Pfaff systems \( P_1 \) and \( P_2 \) associated respectively to \( \Sigma_1 \) and \( \Sigma_2 \)?

j) Integrate the Pfaff systems \( P_1 \) and \( P_2 \), and find the integral manifolds.

k) What are the integral manifolds of the system \( \Sigma_1 \) and those of \( \Sigma_2 \)?

Problem 5.30 (see Chapter 5) is a beautiful application of the Frobenius theorem.

3.30. Exercise. On \( \mathbb{R}^2 \) we consider the following vector field:

\[
\begin{aligned}
x' &= y, \\
y' &= -x + (1 - x^2 - y^2)y.
\end{aligned}
\]

Prove that there is only one closed integral curve.

3.31. Exercise. On \( \mathbb{R}^2 - \{0\} \) find the integral curves of the vector field

\[
\begin{aligned}
x' &= (x^2 + y^2)^{-3/2}(12x^2y^2 - 3x^4 - y^4), \\
y' &= (10xy^3 - 6x^3y)(x^2 + y^2)^{-3/2}.
\end{aligned}
\]

3.32. Exercise. We consider in \( \mathbb{R}^4 \) the Pfaff system \( S \):

\[
\begin{aligned}
\omega^1 &\equiv |x^3 - (x^1)^2|dx^1 + x^1x^2 dx^2 + dx^3 = 0, \\
\omega^2 &\equiv f dx^1 + gdx^2 = 0,
\end{aligned}
\]

where \( \{ x^i \} \) is a coordinate system on \( \mathbb{R}^4 \) and \( f, g \) two functions on \( \mathbb{R}^4 \).

a) Find the functions \( f \) and \( g \) for which the system \( S \) is completely integrable. In this case, show that it is equivalent to a system of rank 2 in \( \mathbb{R}^3 \).

b) What are the integral manifolds of dimension 2 of the system \( S \) when \( f = g = 1 \)?
c) Let us consider the following equation in $\mathbb{R}^3$:

$$\omega \equiv \{x^3 - (x^1)^2 - x^1x^2\}dx^1 \wedge dx^2 - dx^2 \wedge dx^3 + dx^3 \wedge dx^1 = 0.$$ 

Write the integral manifold through the straight line $l(v)$ defined by

$$x^1 = av, \quad x^2 = bv, \quad x^3 = cv,$$

$a, b, c$ being real numbers.

3.33. Exercise. On $\mathbb{R}^2$ we consider the vector field defined by

$$\mathbb{R}^2 \ni (x, y) \longrightarrow X(x, y) = \left( \begin{array}{c} y + x(1 - x^2 - y^2) \\ -x + y(1 - x^2 - y^2) \end{array} \right).$$

a) Show that for any $(x_0, y_0) \in \mathbb{R}^2 \setminus (0, 0)$ this vector field has a unique integral curve $\phi_t(x_0, y_0)$, defined for all $t \in \mathbb{R}$, such that $\phi_0(x_0, y_0) = (x_0, y_0)$.

b) According to the value of $x_0^2 + y_0^2 > 0$, determine whether the image of $\mathbb{R}$ by the map $\psi : t \longrightarrow \phi_t(x_0, y_0)$ is a submanifold of $\mathbb{R}^2$, and whether $\psi$ is an injective immersion or even an imbedding in $\mathbb{R}^2$.

3.34. Problem. On $\mathbb{R}^4$ we consider the following vector field:

$$Y = e^{-2x_1} \frac{\partial}{\partial x^1} - e^{-2x_2} \frac{\partial}{\partial x^2} + e^{-2x_4} \frac{\partial}{\partial x^4}.$$ 

a) Write the Pfaff system (1) associated to $Y$. Find three independent first integrals of system (1): $y^1, y^2, y^3$.

b) Write the equation $Y(f) = 0$ in a new coordinate system $\{y^i\}$ ($i = 1, 2, 3, 4$); $y^4$ is to be chosen. Deduce from this the solutions of $Y(f) = 0$.

c) Let $Z$ be the vector field

$$Z = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \lambda(x^i) \frac{\partial}{\partial x^3} + \mu(x^i) \frac{\partial}{\partial x^4}.$$ 

Write the Pfaff system (2) corresponding to the 2-plane field defined by $Y$ and $Z$ ($\lambda$ and $\mu$ are $C^1$ functions on $\mathbb{R}^4$).

Hint. Integrate an arbitrary vector field which is a linear combination of $Y$ and $Z$.

d) What is the Frobenius condition for system (2) to be completely integrable. Find functions $\lambda$ and $\mu$ satisfying this condition.

Hint. Write system (2) in the form

$$\omega_1 = \lambda \left( e^{2x_1} dx^1 + e^{2x_2} dx^2 \right) - \left( e^{2x_1} + e^{2x_2} \right) dx^3 = 0.$$ 

$$\omega_2 = \mu e^{2x_4} \left( e^{2x_1} dx^1 + e^{2x_2} dx^2 \right) + e^{2(x_1+x_2)} (dx^1 - dx^2) - \left( e^{2x_1} + e^{2x_2} \right) e^{2x_4} dx^4 = 0.$$
e) Verify that this condition is satisfied if we choose
\[ \lambda(x^i) = \phi(x^3)\psi \left( e^{2x^1} + e^{2x^2} \right), \]
where \( \phi \) and \( \psi \) are two arbitrary \( C^1 \) functions of one variable with \( \phi \) never zero, and
\[ \mu(x^i) = e^{2(x^1-x^2)} \quad \text{or} \quad \mu(x^i) = -e^{2(x^2-x^1)}. \]
In these two cases, find the first integrals of system (2). Use these first integrals to solve the system \( Y(f) = Z(f) = 0 \).
f) Let \( W = \partial/\partial x^3 \). We consider the Pfaff system (3) corresponding to the 3-plane field defined by \( Y, Z, W \). What is the Frobenius condition for which system (3) is completely integrable?
h) We choose \( \mu(x^i) \) as previously. Integrate system (3) in the two cases. Find the solutions of the system \( Y(f) = Z(f) = W(f) = 0 \).

Solutions to Exercises and Problems

Solution to Problem 3.23.
a) Set \( A^i_\alpha(u, x) = \partial G^i_\alpha(x)/\partial x^i \). Then \( (G^*\Omega)_{ij}(x) = A^i_\alpha A^j_\beta \Omega_{\alpha\beta}(y) \).
b) According to Proposition 3.6,
\[ \mathcal{L}_{X_u} \Omega = X_u(\Omega_{\alpha\beta})dy^\alpha \wedge dy^\beta + \Omega_{\alpha\beta}(dX^\alpha_u \wedge dy^\beta + dy^\alpha \wedge dX^\beta_u), \]
since \( \mathcal{L}_X d = d\mathcal{L}_X \) and \( \mathcal{L}_X y^\alpha = X^\alpha \). Thus
\[ \mathcal{L}_{X_u} \Omega = (X^\gamma_u \partial_\gamma \Omega_{\alpha\beta} + \Omega_{\gamma\beta} \partial_\alpha X^\gamma_u + \Omega_{\alpha\gamma} \partial_\beta X^\gamma_u) dy^\alpha \wedge dy^\beta, \]
and
\[ G^*_{ul} \mathcal{L}_{X_u} \Omega = A^i_\alpha A^j_\beta (X^\gamma_u \partial_\gamma \Omega_{\alpha\beta} + \Omega_{\gamma\beta} \partial_\alpha X^\gamma_u + \Omega_{\alpha\gamma} \partial_\beta X^\gamma_u) dx^i \wedge dx^j. \]
c) Since
\[ \frac{\partial}{\partial u} (G^*\Omega)_{ij} = \frac{\partial}{\partial u} (A^j_\beta A^i_\alpha)\Omega_{\alpha\beta} + A^i_\alpha A^j_\beta \partial_\gamma X^\gamma_u \partial_\gamma \Omega_{\alpha\beta}, \]
the result follows. Indeed, \( \partial_i = A^i_\alpha \partial_\alpha \).
d) \( X_u(y) \) is a vector field on \( G_u(B) = B_0(u) \). We have
\[ X_u(y) = \frac{\partial G_u(x)}{\partial u} = x = \frac{y}{u}; \]
its components are \( X^\alpha_u(y) = y^\alpha/u \).
e) According to b) and c),
\[ G^*_b \Omega - G^*_a \Omega = \int_a^b \frac{\partial}{\partial u} (G^*\Omega) du = \int_a^b G^*_u \mathcal{L}_{X_u(y)} \Omega du \]
\[ = d \int_a^b G^*_u [i(X_u(y))] \Omega du = d\gamma(a, b). \]
f) We have $A_i^\alpha = \partial G^\alpha_{u}(x)/\partial x^i = u\delta_i^\alpha$ and $(G^\alpha_{a}\Omega)_{ij} = a^2 \delta_i^\alpha \delta_j^\beta \Omega_{\alpha\beta}(y)$. Thus $G^\alpha_{a}\Omega \longrightarrow 0$ when $a \longrightarrow 0$.

g) $G_1^\alpha = \text{Id}$; hence $\Omega = d\gamma$ with $\gamma = \int_0^1 G^\alpha_u [i(X_u(y))\Omega]du$.

**Solution to Problem 3.24.**

a) If $X \neq 0$, there exist $Y_1, Y_2, \cdots, Y_{n-1}$ such that $(X, Y_1, Y_2, \cdots, Y_{n-1})$ is a basis of $T_x(M)$. Thus $\omega(X, Y_1, Y_2, \cdots, Y_{n-1})$ is a sum of terms each having a factor $\Omega(X, Y_i)$ for some $i$. If $\Omega(X, Y) = 0$ for all $Y$, we will have $\omega(X, Y_1, Y_2, \cdots, Y_{n-1}) = 0$, which is in contradiction with the hypothesis.

b) In $T_{x_0}(M)$ we choose $e_1 \neq 0$, then $e_2$ such that $\Omega(e_1, X) = 0$ and $\omega(e_2, X) = 0$. Then $\dim E_1 = 2(m - 1)$. Then in $E_1$ we choose $e_3 \neq 0$ and $e_4$ such that $\omega(e_3, e_4) = 1$. By induction we exhibit the basis $e_i$. Then

$$\Omega(x_0) = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$ 

c) $\omega = P(\Omega_{ij})dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ with $P$ a polynomial in the $\Omega_{ij}$. By hypothesis $P(\Omega_{ij}) \neq 0$ on $V$; we can suppose that $V$ is compact and that the coordinate system exists on $\overline{V}$. Since $P$ is continuous on the $\Omega_{ij}$, there exists $\epsilon > 0$ such that $|a_{ij}| < \epsilon$ for all $(i, j)$ implies $P(\Omega_{ij} + a_{ij}) \neq 0$ on $V$. According to the continuity of $\Omega$, there is a neighbourhood $W$ of $x_0$ such that for $x \in W$ we have

$$|\Omega(x) - \Omega(x_0)|_{ij} | \leq \epsilon$$

for all $(i, j)$. Then the rank of $\Omega_t$ is $2m$ on $W$ for $t \in [0, 1]$. Moreover, since $d(\Omega - \hat{\Omega}) = 0$, there exists $\gamma \in \Lambda^1(V)$ such that $\gamma(x_0) = 0$ and $d\gamma = \Omega - \hat{\Omega}$ (see 3.23).

d) We saw in a) that the kernel of $g : X \longrightarrow i(X)\Omega$ is reduced to $\{0\}$. Thus $g$ is an isomorphism of $T_x(V)$ onto $T_x^* (V)$. We can solve the system $X^i_\gamma \Omega_{ij} = \gamma_j$, $j = 1, 2, \cdots, n$ ($\gamma_j$ being the components of the given differential 1-form $\gamma$), at each point $x \in V$. The solution is a $C^\infty$ function of the $\gamma_j$ and $\Omega_{ij}$, since the determinant $|((\Omega_{ij}))|$ is nowhere zero. So to $\gamma \in \Lambda^1(V)$ there corresponds $X \in \Gamma(V)$. We do the same on the local charts of an atlas. The map $\gamma \longrightarrow X$ is linear.

e) The existence and uniqueness of $f_t(x)$ are given by the Cauchy theorem. Moreover,

$$\frac{\partial}{\partial t} (f_t^* \Omega_t) = \frac{\partial}{\partial t} f_t^* \Omega + t \frac{\partial}{\partial t} f_t^*(\hat{\Omega} - \Omega) + f_t^*(\hat{\Omega} - \Omega).$$

According to 3.23c: we find that

$$\frac{\partial}{\partial t} (f_t^* \Omega_t) = f_t^* [\mathcal{L}_X, \Omega_t + \hat{\Omega} - \Omega].$$
Since $d\Omega_t = 0$, it follows that $L_{X_t}\Omega_t = d[i(X_t)\Omega_t] = d\gamma$. Thus 
\[
\frac{\partial}{\partial t}(f_t^*\Omega_t) = 0.
\]

f) We have $f_t^*\Omega = f_t^*\Omega = \Omega$ according to the previous equality. In the coordinate system corresponding to the local chart $(f_t(W), \Psi(f_1^{-1}))$, 
$\Omega$ is the constant 2-form $\Omega$.

**Solution to Problem 3.25.**

a) We have $d\omega = -2xdx \wedge dy - 2ydz \wedge dx + 2xdy \wedge dz$, and 
$\omega \wedge d\omega = [(1- yz)2x - x(z + x)(-2y) + (1 + xy)(-2x)]dx \wedge dy \wedge dz = 0$.

Thus the equation $\omega = 0$ is completely integrable. Its rank is equal to 1 except on $M$. Indeed, its rank is zero when $yz = 1$, $x(z + x) = 0$, and $xy = -1$. The second equation gives $z = -x$, since $x = 0$ is impossible according to the third equation. So the rank is zero on the differential curve $M$ defined by the equations $z = -x$ and $yz = 1$.

If the rank of the map $\Psi : (x, y, z) \rightarrow (f_1, f_2)$ with $f_1 = z + x$ and $f_2 = yz - 1$ is two on $M$, then $M$ is a submanifold of $\mathbb{R}^3$. Now the rank of $D\Psi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is two on $M$, since $z = 0$ and $y = 0$ do not happen simultaneously on $M$.

b) $\omega(Y) = a(1-yz) + c(1+xy)$. We can choose $a = 1+xy$ and $c = yz-1$.

Now we have to consider the system
\[
\frac{dx}{dt} = 1 + xy, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = yz - 1.
\]

Integrating yields $y = y_0$, $\log |1 + xy| = y_0 t + \text{Constant}$ and $\log |yz_0 - 1| = y_0 t + \text{Constant}$. So if $y_0 \neq 0$, the integral curves of the vector field $Y$ are given by
\[
x = \frac{1}{y_0} (\mu e^{yt} - 1), \quad y = y_0, \quad z = \frac{1}{y_0} (\lambda e^{yt} + 1),
\]

$\mu = (x_0y_0 + 1)$ and $\lambda = (y_0z_0 - 1)$. They are straight lines in the planes $y = y_0$. If $y_0 = 0$, the integral curves are straight lines
\[
\begin{cases}
x + z = \nu, \\
y = 0,
\end{cases}
\]

where $\lambda, \mu, \nu$ are constants.

c) We have
\[
\frac{dx}{1 + xy_0} = \frac{dz}{yz_0 - 1}.
\]

Thus the second first integral may be
\[
u = \frac{zy - 1}{xy + 1}.
\]
Putting $z = ux + (u + 1)/y$ in the expression of $\omega$ yields

$$\omega = -u(1 + xy)dx - x\left(x + ux + \frac{u + 1}{y}\right)dy$$

$$+(1 + xy)\left[udx + xdu + \frac{du}{y} - (u + 1)\frac{dy}{y^2}\right],$$

$$\omega = (1 + xy)^2y^{-2}[-(u + 1)dy + ydu].$$

We could expect this result. As $u$ and $y$ are first integrals, we have $dy(Y) = 0$ and $du(Y) = 0$.

So $\omega(Y) = 0$ and $dx(Y) \neq 0$ imply that $dx$ cannot appear in the expression of $\omega$ in the coordinate system $(x, y, u)$.

d) We can write $\omega = (1 + xy)(z + x)[du/(1 + u) - dy/y]$.

If $(1 + xy)(z + x) \neq 0$, then $\omega = 0$ is equivalent to

$$\tilde{\omega} = \frac{du}{1 + u} - \frac{dy}{y} = 0.$$

$\tilde{\omega} = 0$ leads to $1 + xy = k(z + x)$, with $k \in \mathbb{R}$. They are the integral manifolds of the equation $\omega = 0$. Indeed, we verify that $1 + xy = 0$ ($k = 0$) is an integral manifold (when $x = -\frac{1}{y}, \omega = 0$). The plane $z + x = 0$ is also an integral manifold. For $k \in \mathbb{R} - \{0\}$, we have to consider the map

$$\varphi : (x, y, z) \rightarrow 1 + xy - k(z + x),$$

$D\varphi = (y - k, x, -k)$, which is of rank 1, except for $k = 0$ if $x = y = 0$. Hence the integral manifolds are submanifolds of $\mathbb{R}^3$.

Through a point $(x_0, y_0, z_0)$ of $\mathbb{R}^3 - M$, there is a unique integral manifold, the one with $k = (1 + x_0y_0)/(z_0 + x_0)$.

Through a point of $M$, there are two integral manifolds: the plane $x + z = 0$ and the surface $1 + xy = 0$.

e) If $X$ is of the form $X = b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$ with $b = 1 + xy$ and $c = x(z + x)$, then $\omega(x) = 0$. Integrating the equation

$$\frac{xdy}{1 + xy} = \frac{dz}{z + x},$$

with $x = x_0$ gives at once the first integral

$$v = \frac{1 + xy}{z + x},$$
Solutions to Exercises and Problems

Solution to Problem 3.26.

a) A hyperplane $H$ is defined by a vector orthogonal to $H$. Let $\{a_i\}$ ($1 \leq i \leq n+1$) be its components. If the hyperplane $H$ is not parallel to $Oz$, then $a_{n+1} \neq 0$. Then we choose, as vector orthogonal to $H$, the vector having a bijection between the hyperplanes which are not parallel to $Oz$ and the set $\{a_1, a_2, \ldots, a_n\}$, which may be considered as the coordinates of a point of $E^*$. 

b) The $df_i = (\partial f_i/\partial x_j)dx^j + (\partial f_i/\partial z)dz$ ($1 \leq i \leq q$) are independent. A regular contact element will be tangent to $W$ if $\theta = a_i dx^i + dz$ is a linear combination of the $df_i$.

Suppose that the determinant $|D| = |\partial f_i/\partial x_j|$ ($1 \leq i \leq q$, $1 \leq j \leq q$) does not vanish. Then, for every $k > q$,

$$D_k = \begin{vmatrix} D & \frac{\partial f_1}{\partial x_k} \\ \vdots \\ a_1 & \cdots & a_q & a_k \end{vmatrix} = 0 \quad \text{and} \quad \hat{D} = \begin{vmatrix} D & \frac{\partial f_1}{\partial z} \\ \vdots \\ a_1 & \cdots & a_q & 1 \end{vmatrix} = 0.$$

$W$ belongs to the space $F \times E^*$, which has dimension $2n+1$. But among the $n+1$ first coordinates only $p = n + 1 - q$ are independent: $(x, z) \in W$. For the $n$ last coordinates, we have $p$ independent conditions. Indeed, $\partial D_k/\partial a_k = |D| \neq 0$, and $a_k$ ($k > q$) appears only once in the determinants $D_j$ (when $j = k$). Moreover, $\hat{D} = |D| + \sum_{1 \leq i \leq q} \lambda^i a_i = 0$.

Since $|D| \neq 0$, at least one of the $\lambda^i$ ($1 \leq i \leq q$) is not zero, say $\lambda^j \neq 0$. In the Jacobian matrix of

$$(a_1, a_2, \ldots, a_n) \rightarrow (D_{q+1}, \ldots, D_n, \hat{D}),$$

we can exhibit the determinant

$$\begin{vmatrix} |D| & 0 & \frac{\partial D_k}{\partial a_j} \\ |D| & \vdots & \vdots \\ 0 & \lambda^j \end{vmatrix},$$

which does not vanish.

The dimension of $\hat{W}$ is $(p + n) - p = n$.

c) When $p = n$, at a point $(x, z) \in W$ there is only one contact hyperplane, defined by $a_j = \partial z/\partial x^j$. Thus $dz = a_j dx^j$. When $p < n$, $W$ is defined by $z = f_i(x)$ ($1 \leq i \leq q$) at least when the contact element is regular. And $a_j = \lambda^i \partial f_i/\partial x^j$ for some $\lambda^i$ with $\sum \lambda^i = 1$.

Thus we verify that $\omega = 0$ on $\hat{W}$. 

d) The answer is no. Otherwise the integral manifold depends on $d$ variables. For instance we would have $z = f(x^1, \ldots, x^n, a_{\lambda_1}, \ldots, a_{\lambda_j})$, and $dz$ would be expressed as a function of the $dx^j$ but also as a function of some $da_j$, which is not the case.
Chapter 4

Linear Connections

In $\mathbb{R}^n$, there is a natural notion: the parallel transport of a geometric picture. This allows one to compare two vectors in $\mathbb{R}^n$ which do not have the same origin. On a manifold, that notion does not exist—it is impossible to compare two vectors which do not belong to the same tangent space. Closely related to this is the fact that if we consider a vector field $Y$, we do not know what it means to differentiate $Y$ at a point in a direction. Henceforth, we assume that the manifold is $C^\infty$ differentiable.

First Definitions

4.1. In the preceding chapter, thanks to the linear tangent map $(\varphi_{-t})_\ast$, we transported the vector $Y_{\varphi_{-t}(P)}$ into the tangent space $T_P(M)$, and we defined $\mathcal{L}_XY$. But, as we saw, $\mathcal{L}_XY = [X, Y]$ depends not only on $X_P \in T_P(M)$, but also on the vector field $X$.

This is the reason for introducing a connection on a manifold. We define below the derivative of a vector field (then, more generally, of a tensor field) at a point in a direction.

4.2. Definition. A connection on a differentiable manifold $M$ is a mapping $D$ (called the covariant derivative) of $T(M) \times \Gamma(M)$ into $T(M)$ which has the following properties:

a) If $X \in T_P(M)$, then $D(X, Y)$ (denoted by $D_X Y$) is in $T_P(M)$.

b) For any $P \in M$ the restriction of $D$ to $T_P(M) \times \Gamma(M)$ is bilinear.

$D_X(fY) = X(f)Y + fD_X Y$.
d) If \( X \) and \( Y \) belong to \( \Gamma(M) \), \( X \) of class \( C^r \) and \( Y \) of class \( C^{r+1} \), then 
\[ D_X Y \] 
is in \( \Gamma(M) \) and is of class \( C^r \).

Recall that \( \Gamma(M) \) denotes the space of differential vector fields (2.14).

A natural question arises: On a given \( C^\infty \) differentiable manifold, does there exist a connection? The answer is yes, there does. We will prove in Chapter 5 that a particular connection, the Riemannian connection, exists. So we will be able to differentiate a vector field with respect to a given vector. Then, applying Proposition 4.5, we have all connections.

Let us write the covariant derivative \( D_X Y \) of a vector field \( Y \) with respect to a vector \( X \in T_P(M) \) in a local coordinate system \( \{x^i\} \) corresponding to a local chart \( (\Omega, \varphi) \) with \( P \in \Omega \):

\[
X = X^i \frac{\partial}{\partial x^i} \quad \text{and} \quad Y = Y^j \frac{\partial}{\partial x^j}.
\]

\( \{\partial/\partial x^i\} \) \((i = 1, 2, \ldots, n)\) are \( n \) vector fields on \( \Omega \) which form at each point \( Q \in \Omega \) a basis of \( T_Q(M) \), as we saw in Chapter 2. \( \{X^i\} \) \((1 \leq i \leq n)\) and \( \{Y^j\} \) \((1 \leq j \leq n)\) are the components of \( X \in T_P(M) \) and \( Y \in \Gamma(\Omega) \).

According to (c) and the bilinearity of \( D \),

\[
D_X Y = X^i D_i Y = X^i (\partial_i Y^j) \frac{\partial}{\partial x^j} + X^i Y^j D_i \left( \frac{\partial}{\partial x^j} \right),
\]

where we denote \( D_{\partial_i/\partial x^i} \) by \( D_i \) and \( \partial f/\partial x^i \) by \( \partial_i f \), to simplify the notation.

According to (d), \( D_i (\partial/\partial x^j) \) is a vector field on \( \Omega \). Writing in the basis \( \{\partial/\partial x^k\} \),

\[
D_i \left( \frac{\partial}{\partial x^j} \right) = \Gamma^k_{ij} \frac{\partial}{\partial x^k}.
\]

Christoffel Symbols

4.3. Definition. \( \Gamma^k_{ij} \) are called Christoffel symbols of the connection \( D \) with respect to the local coordinate system \( x^1, x^2, \ldots, x^n \). They are \( C^\infty \) functions in \( \Omega \), according to d) (the manifold is assumed to be \( C^\infty \)). They define the local expression of the connection in the local chart \( (\Omega, \varphi) \). Conversely, if for all pairs \((i, j)\) we are given \( D_i (\partial/\partial x^j) = \Gamma^k_{ij} \partial/\partial x^k \), then a unique connection \( D \) is defined in \( \Omega \).

4.4. Definition. \( \nabla Y \) is the differential \((1,1)\)-tensor which in the local chart \( (\Omega, \varphi) \) has \( (D_i Y)^j \) as components. \( (D_i Y)^j \) is the \( j^{th} \) component of the vector field \( D_i Y \).

To simplify we write \( \nabla_i Y^j \) instead of \( (\nabla Y)^j_i \). According to the definition above, \( (D_i Y)^j \) is equal to \( \nabla_i Y^j \):

\[
\nabla_i Y^j = \partial_i Y^j + \Gamma^j_{ik} Y^k
\]
4.5. Proposition. The Christoffel symbols $\Gamma^i_{ij}$ of a connection $D$ are not the components of a tensor field. If $\tilde{\Gamma}^i_{ij}$ are the Christoffel symbols of another connection $\tilde{D}$, then $C^i_{ij} = \tilde{\Gamma}^i_{ij} - \Gamma^i_{ij}$ are the components of a $(2,1)$-tensor, twice covariant and once contravariant.

Conversely, if $\Gamma^i_{ij}$ are the Christoffel symbols of a connection $D$ and $C^i_{ij}$ the components of a $(2,1)$-tensor field, then $\tilde{\Gamma}^i_{ij} = C^i_{ij} + \Gamma^i_{ij}$ are the Christoffel symbols of a connection.

**Proof.** Let $(\theta, \psi)$ be another local chart at $P$, $\{y^\alpha\}$ the associated coordinate system. Let $A^\alpha_i = \partial y^\alpha / \partial x^i$ and $B^\beta_j = \partial x^j / \partial y^\beta$. If $X = X^i \partial / \partial x^i = X^\alpha \partial / \partial y^\alpha$, $Y = Y^j \partial / \partial x^j = Y^\beta \partial / \partial y^\beta$, and $\omega = \omega_k dx^k = \omega_\lambda dy^\lambda$ is a differential 1-form, then

$$X^i (\nabla_i Y^j) \omega_j = X^\alpha (\nabla_\alpha Y^\beta) \omega_\beta = X^i A^\alpha_i [\partial_\alpha (Y^k A^\beta_k) + \Gamma^\beta_\alpha A^\lambda_k Y^k]B^\beta_j \omega_j.$$  

$X^i (\nabla_i Y^j) \omega_j$ is a real function, so it is the same in any chart. Moreover, we can write $X^\alpha = A^\alpha_i x^i$, $Y^\beta = A^\beta_k y^k$ and $\omega_\beta = B^\beta_j \omega_j$. Since the equality above is true for any $X$ and $\omega$,

$$\partial_i Y^j + \Gamma^j_{ik} Y^k = A^\alpha_i [(\partial_\alpha Y^k) A^\beta_k + Y^k \partial_\alpha A^\beta_k + \Gamma^\beta_\alpha A^\lambda_k Y^k]B^\beta_j.$$  

As $\partial_i Y^j = A^\alpha_i \partial_\alpha Y^j$, $A^\beta_k B^\beta_j = \delta^j_k$ ($= 0$ if $k \neq j$; $= 1$ if $k = j$) and since the equality is true for any $Y$, we get

$$\Gamma^j_{ik} = B^\beta_j \partial_i A^\beta_k + A^\alpha_i A^\lambda_k B^\beta_j \Gamma^\beta_\alpha.$$  

Christoffel symbols $\Gamma^j_{ik}$ are not the components of a tensor; in a change of local chart they do not transform like the components of a tensor, whereas $C^j_{ik}$ are the components of a tensor. Indeed, since $\tilde{\Gamma}^j_{ik} = B^\beta_j \partial_i A^\beta_k + A^\alpha_i A^\lambda_k B^\beta_j \tilde{\Gamma}^\beta_\alpha$, we get $C^j_{ik} = A^\alpha_i A^\lambda_k B^\beta_j C^\beta_\alpha$.

Conversely, in $\Omega$, the family of $C^\infty$ functions $C^i_{ij} + \Gamma^i_{ij}$ defines a connection $\tilde{D}$. Indeed, the mapping

$$\tilde{D} : (X, Y) \rightarrow D_X Y + C^j_{ik} X^i Y^k \frac{\partial}{\partial x^j}$$  

satisfies conditions (a), (b), (c) and (d) of Definition 4.2.

**Torsion and Curvature**

4.6. Definition. The torsion of a connection $D$ is the map of $\Gamma \times \Gamma$ into $\Gamma$ defined by

$$(X, Y) \rightarrow T(X, Y) = D_X Y - D_Y X - [X, Y].$$
Let us verify that the value of $T(X, Y)$ at $P$ depends only on the values of $X$ and $Y$ at $P$, and does not depend on the first derivatives of the components of $X$ and $Y$. We have

$$[T(X, Y)]^i = X^j \nabla_j Y^i - Y^j \nabla_j X^i - (X^j \partial_j Y^i - Y^j \partial_j X^i)$$

$$= (\Gamma^i_{jk} - \Gamma^i_{kj}) X^j Y^k = T^i_{jk} X^j Y^k.$$ 

The operator of $T_P(M) \times T_P(M)$ into $T_P(M)$ defined by

$$T : (X_P, Y_P) \mapsto T^i_{jk}(P) X^i_P Y^j_P \left( \frac{\partial}{\partial x^k}\right)_P$$

is represented by $T^i_{jk}(P)$, which are the components of a $(2,1)$-tensor since contraction with two vectors yields a vector. In $\Omega$, the $T^i_{jk}$ are $C^\infty$ functions.

It is possible to give an alternative proof. Since $\delta_i A^\beta_k = \delta_{ik} y^\beta = \delta_{ki} y^\beta = \delta_k A^\beta_i$, according to the formula in 4.5 with $\Gamma^i_{jk}$ and $\Gamma^j_{ki}$ we get

$$T^i_{jk} = A^\alpha_i A^\beta_k B^\gamma_{\alpha\beta} \delta^\gamma.$$ 

So the $T^i_{jk}$ are the components of a $(2,1)$-tensor.

4.7. Definition. The curvature of a connection $D$ is a 2-form with values in $\text{Hom}(\Gamma, \Gamma)$ defined by

$$(X, Y) \mapsto R(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$ 

For the definition we suppose that the vector fields are at least $C^2$. One verifies that $R(X, Y)Z$ at $P$ depends only on the values of $X, Y$ and $Z$ at $P$. In a local chart,

$$[R(X, Y)Z]^k = X^i D_i [Y^j (D_j Z)^k] - Y^i D_i [X^j (D_j Z)^k]$$

$$- (X^i \partial_i Y^j - Y^i \partial_i X^j)(D_j Z)^k$$

$$= (X^i \partial_i Y^j - Y^i \partial_i X^j - [X, Y]^j)(D_j Z)^k$$

$$+ X^i Y^j [\partial_i (\partial_j Z^k + \Gamma^k_{ji} Z^l) + \Gamma^k_{il} (D_j Z)^l]$$

$$- X^i Y^j [\partial_j (\partial_i Z^k + \Gamma^k_{ij} Z^l) + \Gamma^k_{lj} (D_i Z)^l].$$

The terms in $(D_j Z)^k, \partial_i Z^l$ and $\partial_j Z^l$ vanish. Thus we get

$$[R(X, Y)Z]^k = X^i Y^j Z^m (\partial_i \Gamma^k_{jm} - \partial_j \Gamma^k_{im} + \Gamma^k_{il} \Gamma^l_{jm} - \Gamma^k_{jl} \Gamma^l_{im}).$$

Denote by $R^k_{lij}$ the $k^{th}$ component of $R(\partial/\partial x^i, \partial/\partial x^j) \partial/\partial x^l$. We have

$$R^k_{lij} = \delta_i \Gamma^k_{jl} - \delta_j \Gamma^k_{il} + \Gamma^k_{im} \Gamma^m_{jl} - \Gamma^k_{jm} \Gamma^m_{il}.$$ 

$R^k_{lij}$ are the components of a $(3,1)$-tensor on $\Omega$, since after contraction with three vectors we get the components of a vector $R^k_{lij} Z^l X^i Y^j = [R(X, Y)Z]^k$. 


According to the definition, $R(X, Y) = -R(Y, X)$; thus $R^k_{ij} = -R^k_{ji}$. Since $[\partial / \partial x^i, \partial / \partial x^j] = 0$, if we choose $X = \partial / \partial x^i$ and $Y = \partial / \partial x^j$ in $\Omega$, we get

$$[(D_i D_j Z)^k - (D_j D_i Z)^k] \frac{\partial}{\partial x^k} = R \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) Z = R^k_{ij} Z^l \frac{\partial}{\partial x^k}.$$  

**Parallel Transport. Geodesics**

4.8. Definition. A vector field $X$ is said to be parallel along a differentiable curve $C$ if its covariant derivative in the direction of the tangent vector to $C$ is zero. Letting $X(t) = X(C(t))$, we get

$$D_{C(t)} X(t) = \frac{dC^i(t)}{dt} \nabla_i X(t) = \frac{dC^i(t)}{dt} [\partial_i X^j(t) + \Gamma^j_{ik}(C(t)) X^k(t)] \frac{\partial}{\partial x^j} = 0.$$  

Thus $X(t)$ is a parallel vector field along $C$ if, in a local chart,

$$\frac{dX^i(t)}{dt} + \Gamma^i_{jk}(C(t)) X^k(t) \frac{dC^j(t)}{dt} = 0.$$  

One verifies that if two vector fields are equal to each other at each point of $C$, then one is parallel along $C$ if and only if the other is. Thus, for a vector field $X$ to be parallel along $C$ depends only on the values of $X$ along $C$. This is why we will talk about parallel vector fields along $C$, even if they are defined only along $C$.

4.9. Definition. Let $P$ and $Q$ be two points of $M$, $C(t)$ a differentiable curve from $P$ to $Q$ ($C(a) = P, C(b) = Q$), and $X_0$ a vector of $T_P(M)$. According to Cauchy's theorem, the initial value problem $X(a) = X_0$ of the equation above has a unique solution $X(t)$ defined for all $t \in [a, b]$, since the equation is linear. The vector $X(b)$ of this parallel vector field along $C$ (with $X(a) = X_0$) is called the parallel translated vector of $X_0$ from $P$ to $Q$ along $C$.

The parallel translate along a piecewise differentiable curve $C = \bigcup_{i=1}^m C_i$ is defined in a natural way. $X_1$ is the parallel translated vector of $X_0$ along the differentiable curve $C_1$, and by induction $X_i$ is the parallel translated vector of $X_{i-1}$ along the differentiable curve $C_i$. $X_m$ is called the parallel translated vector of $X_0$ along $C$. The definition of the parallel translate of a tensor along $C$ is similar, once we define the covariant derivative of a tensor field.

The parallel translate at $Q$ of $X_0$ along $C$ is unique according to Cauchy's theorem, but it depends on $C$. That is not the case in $\mathbb{R}^n$. For instance, on $S_n$, consider a unit tangent vector $X_0$ at the north pole $P$, and let $C$ the following piecewise differentiable curve: half meridian tangent to $X_0$, then
a piece of equator \( \hat{AB} \), and finally another half meridian back to \( P \). Any given unit tangent vector at \( P \) may be the parallel translated vector of \( X_0 \) along \( C \); it depends on \( B \). It is possible to verify this result after Chapter 5 (Exercises 5.36 and 5.51).

The parallel translated vector of \( X_0 \) along \( C \) (resp. \( \hat{PA} \)) is \( X_3 \) (resp. \( X_1 \)); \( X_2 \) is the parallel translated vector of \( X_1 \) along \( \hat{AB} \).

**4.10. Definition.** A differentiable curve \( C(t) \) of class \( C^2 \) is a geodesic if its field of tangent vectors is parallel along \( C(t) \). That is to say, \( D_{dC/dt} \frac{dC}{dt} = 0 \). Writing in a local chart, we find that \( C(t) \) is a geodesic if and only if

\[
\frac{d^2 C^j(t)}{dt^2} + \Gamma^j_{ik}(C(t)) \frac{dC^i(t)}{dt} \frac{dC^k(t)}{dt} = 0.
\]

This is the geodesic equation. It is obtained from the parallel translation equation (see 4.8) with \( X(t) = dC(t)/dt \). But this equation is not linear—it is much more complicated. We will study it in detail in Chapter 5. It is an equation of the second order. According to Cauchy's theorem,

**4.11. Proposition.** Given \( P \in M \) and \( X \in T_P(M) \), there exists a unique geodesic, starting at \( P \), such that \( X \) is its tangent vector at \( P \). This geodesic depends smoothly on the initial conditions at \( P \) and \( X \). If \( X = 0 \), the geodesic is reduced to the point \( P \).

**4.12. Example.** Connections on an open set \( \Omega \subset \mathbb{R}^n \) (recall that \( \Omega \) is a manifold). There are atlases with one chart, for instance \( (\Omega, \operatorname{Id}) \). If on \( (\Omega, \operatorname{Id}) \) we choose arbitrary Christoffel symbols, they define a connection. When there is more than one chart in the atlas, this procedure cannot be applied according to Proposition 4.5.

If on \( (\Omega, \operatorname{Id}) \) we choose \( \Gamma^k_{ij} = 0 \), the curvature vanishes. The parallel transport is the usual one. Indeed, the equation (see 4.3) is then
\[ \frac{dX^j(t)}{dt} = 0, \] so the components of \( X(t) \) are constant. The geodesic equation is then \( \frac{d^2C^j(t)}{dt^2} = 0 \), and so geodesics are straight lines (each \( C^j(t) \) is a linear function of \( t \)).

**Covariant Derivative**

4.13. **Definition.** The definition of covariant derivative extends to differentiable tensor fields as follows:

a) For functions, \( D_X f = X(f) \).

b) \( D_X \) preserves the type of the tensor.

c) \( D_X (u \otimes v) = (D_X u) \otimes v + u \otimes (D_X v) \), where \( u \) and \( v \) are tensor fields.

d) \( D_X \) commutes with the contraction.

Let us show how we compute the covariant derivative of a differential 1-form \( \omega \) in a local chart. Let \( X = X^i \partial / \partial x^i \), \( Y = Y^j \partial / \partial x^j \) and \( \omega = \omega_k dx^k \).

According to (a)
\[
D_X (\omega_k Y^k) = X^i \partial_i (\omega_k Y^k),
\]
according to (c)
\[
D_X (\omega \otimes Y) = (D_X \omega) \otimes Y + \omega \otimes (D_X Y),
\]
and according to (d)
\[
D_X (\omega_k Y^k) = D_X [\omega(Y)] = (D_X \omega)(Y) + \omega(D_X Y).
\]
Thus
\[
(D_X \omega)_k Y^k = X^i (\partial_i \omega_k) Y^k + X^i \omega_k \partial_i Y^k - \omega_k X^i \nabla_i Y^k.
\]
Setting \( X = \partial / \partial x^i \) leads to
\[
(D_i \omega)_k Y^k = (\partial_i \omega_k) Y^k - \omega_k \Gamma^k_{ij} Y^j.
\]
Interchanging the dummy indices \( k \) and \( j \) in the last term yields, since the equality is true for any \( Y \),
\[
(D_i \omega)_k = \partial_i \omega_k - \Gamma^j_{ik} \omega_j.
\]
By definition, \( \nabla \omega \) is the twice covariant tensor having components \( (\nabla \omega)_{ik} \) written \( \nabla_i \omega_k \) equal to \( (D_i \omega)_k \):
\[
\nabla_i \omega_k = \partial_i \omega_k - \Gamma^j_{ik} \omega_j.
\]
In a similar way we compute the covariant derivative \( \nabla_i \) of an \((r, s)\)-tensor field \( u \).

**The rule is the following:** \( \nabla_i u \) is the sum of the partial derivative \( \partial_i u \), of \( s \) terms with \( \Gamma^j_{ik} \), where \( j \) is equal successively to the values of the \( s \) contravariant indices and \( k \) is a dummy index, and \( r \) terms with \( -\Gamma^j_{ik} \).
where \( j \) is a dummy index and \( k \) is equal successively to the values of the \( r \) covariant indices.

4.14. Examples. Let \( g \) be a \((2,0)\)-tensor field. We have

\[
\nabla_i g_{jk} = \delta_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl}.
\]

Let \( R^k_{ij} \) be the components of the curvature tensor. Its covariate derivative is

\[
\nabla_m R^k_{ij} = \delta_m R^k_{ij} - \Gamma^\alpha_{ml} R^\alpha_{ij} + \Gamma^k_{ml} R^\alpha_{ij} - \Gamma^\alpha_{ml} R^k_{\alpha j} - \Gamma^\alpha_{mj} R^l_{\alpha i}.
\]

4.15. Exercise. Prove the following formulas (commuting of the covariant derivatives):

\[
\nabla_i \nabla_j Z^k - \nabla_j \nabla_i Z^k = R^k_{lij} Z^l - T^l_{ij} \nabla_l Z^k
\]

and

\[
\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k = -R^l_{kij} \omega_l - T^l_{ij} \nabla_l \omega_k.
\]

The first formula looks strange when we notice that, by Definition 4.7, \((D_i D_j Z)^k - (D_j D_i Z)^k = R^k_{lij} Z^l\).

The difference comes from the fact that \( D_j Z \) is a vector field whereas \( \nabla Z \) is a \((1,1)\)-tensor field,

\[
(D_i D_j Z)^k = \delta_i (D_j Z)^k + \Gamma^k_{il} (D_j Z)^l,
\]

whereas

\[
\nabla_i \nabla_j Z^k = \delta_i (\nabla_j Z^k) + \Gamma^k_{il} \nabla_j Z^l - \Gamma^l_{ij} \nabla_l Z^k.
\]

Thus

\[
\nabla_i \nabla_j Z^k - \nabla_j \nabla_i Z^k = (D_i D_j Z)^k - (D_j D_i Z)^k - T^l_{ij} \nabla_l Z^k,
\]

and the first result follows. For the second result, we can compute \( A = \nabla_i \nabla_j (Z^k \omega_k) - \nabla_j \nabla_i (Z^k \omega_k) \) in two different ways:

\[
A = \delta_{ij} (Z^k \omega_k) - \Gamma^l_{ij} \delta_l (Z^k \omega_k) - \delta_{ji} (Z^k \omega_k) + \Gamma^l_{ji} \delta_l (Z^k \omega_k)
\]

\[
= -T^l_{ij} \nabla_l (Z^k \omega_k),
\]

since for functions \( \delta_i f = \nabla_i f \). And if we develop \( A \) as

\[
A = \omega_k (\nabla_i \nabla_j Z^k - \nabla_j \nabla_i Z^k) + Z^k (\nabla_i \nabla_j \omega_k - \nabla_j \nabla_i \omega_k),
\]

the second formula follows.

4.16. Proposition. The Bianchi identities are

\[
\sum_{\sigma(i,j,k)} R^l_{ijk} = \sum_{\sigma(i,j,k)} (\nabla_j T^l_{ki} - T^m_{ji} T^l_{mk})
\]

and

\[
\sum_{\sigma(i,j,k)} \nabla_j R^m_{lki} = \sum_{\sigma(i,j,k)} T^A_{lk} R^m_{ihj}.
\]
where \( \sum_{\sigma(i,j,k)} \) means the sum on the circular permutations of \( i, j \) and \( k \).

When the connection is torsion free, the Bianchi identities are simple. The first identity is \( \sum_{\sigma(i,j,k)} R^i_{jk} = 0 \) and the second \( \sum_{\sigma(i,j,k)} \nabla_j R^m_{ik} = 0 \). We will prove them in Chapter 5.

Exercises and Problems

4.17. Exercise. Consider on \( \mathbb{R}^2 \) the connection defined by \( \Gamma^1_{11} = x^1, \Gamma^1_{12} = 1, \Gamma^2_{22} = 2x^2 \), the other Christoffel symbols vanishing. Let \( C \) be the arc on \( \mathbb{R}^2 \) defined by \( C^1(t) = t, C^2(t) = 0, t \in [0, 1] \).

a) Compute the parallel translate along \( C \) of the tangent vector \( \partial/\partial x^2 \) at \( O \).

b) Write the geodesic equation of this connection.

4.18. Exercise. Consider a differentiable manifold endowed with a connection.

a) Give, in local coordinates, the expression of \( D_0 1_{0} (dx^i) \) and of \( DX T \), where \( T \) is a \((2,1)\)-tensor field, by means of the Christoffel symbols.

b) Write the differential equation satisfied by a \((1,1)\)-tensor field parallel along a differentiable curve \( C \).

4.19. Problem. Let \( M_n \) \((n > 1)\) be a \( C^\infty \) differentiable manifold endowed with a linear connection (\( \Gamma^k_{ij} \) are the Christoffel symbols in a coordinate system).

a) We say that two vector fields that are never zero have the same direction, if at each point of \( M \) the vectors of these vector fields have the same direction. So we define an equivalence relation in the set of vector fields that do not vanish.

A differentiable curve \( C(s) \) being given, we say that the vector field \( X \) preserves a parallel direction along \( C \) if there exists in the equivalence class of \( X \) a parallel vector field along \( C \).

Prove that this property is equivalent to the condition that

\[
\frac{dX^k(s)}{ds} + \Gamma^k_{ij}(C(s))X^j(s)\frac{dC^i(s)}{ds} \quad \text{and} \quad X
\]

have the same direction.

Write the differential system satisfied by the curve \( C \) whose tangent vectors preserve a parallel direction along \( C \). Show that these curves \( C \) are geodesics up to a change of parameters.

b) We consider on \( M \) a second linear connection whose Christoffel symbols are \( \tilde{\Gamma}^k_{ij} \), and we set \( T^k_{ij} = \tilde{\Gamma}^k_{ij} - \Gamma^k_{ij} \). Determine the \( T^k_{ij} \) so that the two connections define the same parallelism: for any differentiable
curve, if a vector field preserves a parallel direction for one connection, it does the same for the other.

Deduce from the above result that the necessary and sufficient condition we are looking for is \( T_{ij}^k = \mu_i \delta_j^k \), where the \( \mu_i \) are the components of a differentiable 1-form.

c) Given a linear connection on \( M_n \), is it possible to exhibit a connection without torsion which defines the same parallelism?

4.20. Exercise. Let \( M \) be a differentiable manifold and let \( D, \tilde{D} \) be two connections on \( M \). For two vector fields \( X \) and \( Y \) we set

\[
A(X, Y) = \tilde{D}XY - DXY \quad \text{and} \quad B(X, Y) = A(X, Y) + A(Y, X).
\]

a) Show that a necessary and sufficient condition for \( D \) and \( \tilde{D} \) to have the same geodesics is that \( B(X, Y) = 0 \).

b) Verify that \( D \) and \( \tilde{D} \) are identical if the two connections have the same geodesics and torsion.

c) Prove that for \( D \) and \( \tilde{D} \) to have the same geodesics up to a change of parameterization, it is necessary and sufficient that \( A(X, X) \) be proportional to \( X \).

d) In this case exhibit a 1-form \( \omega \) such that

\[
B(X, Y) = \omega(X)Y + \omega(Y)X.
\]

Solutions to Exercises

Solution to Exercise 4.17.

a) Let \( X^i(t) = X^i(C(t)) \) be the components of the parallel vector field \( X \) along \( C \) which is equal to \( \partial/\partial x^2 \) at 0. Then

\[
\frac{dX^1}{dt} + \Gamma^1_{jk}(C(t)) \frac{dC^j}{dt} X^k = \frac{dX^1}{dt} + x^1 X^1 + X^2 = 0,
\]

\[
\frac{dX^2}{dt} = 0.
\]

Thus \( X^2 = 1 \) and \( (X^1)' + x^1 X^1 + 1 = 0 \) with \( X^1(0) = 0 \).

The general solution is

\[
X^1(t) = -e^{-\frac{t^2}{2}} \int_0^t e^{\frac{u^2}{2}} du + ke^{-\frac{t^2}{2}}.
\]

Also, \( X^1(0) \) implies \( k = 0 \). So

\[
X^1 = \frac{-1}{\sqrt{e}} \int_0^1 e^{\frac{u^2}{2}} du \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}.
\]
b) We have

\[
\frac{d^2 C^1}{dt^2} + C^1 \left( \frac{dC^1}{dt} \right)^2 + \frac{dC^1}{dt} \frac{dC^2}{dt} = 0,
\]

\[
\frac{d^2 C^2}{dt^2} + 2C^2 \left( \frac{dC^2}{dt} \right)^2 = 0.
\]

Solution to Exercise 4.18.

a) \(D_{\partial/\partial x^i}(\omega)\), with \(\omega = dx^j\), is a differential 1-form. Also, \(\omega = \omega_k dx^k\), with \(\omega_j = 1\) and \(\omega_k = 0\) if \(k \neq j\). Thus \(D_{\partial/\partial x^i}(dx^j) = -\Gamma^j_{ik} dx^k\). Finally,

\[
DXT = X^l [\partial_i T^k_{ij} - \Gamma^m_{li} T^k_{mj} - \Gamma^m_{lj} T^k_{im} + \Gamma^k_{lm} T^m_{ij}] dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}.
\]

b) The equation is \(D_{dC/dt} Z = 0\) with \(Z = Z^i dx^i \otimes \partial/\partial x^i\). That is, for any \((i,j)\),

\[
\frac{dZ^j_i}{dt} + \Gamma^j_{kl} \frac{dC^k_i}{dt} Z^l_i - \Gamma^l_{ki} \frac{dC^k_i}{dt} Z^j_i = 0.
\]
Riemannian Manifolds

In this chapter we will apply what we learned in the previous chapters to the study of Riemannian manifolds. When we have to study a manifold, it is convenient to consider a Riemannian metric on it. We can do this without loss of generality, since there exists a $C^\infty$ Riemannian metric $g$ on a $C^\infty$ paracompact differentiable manifold $M$. We define a distance on $(M, g)$, so $(M, g)$ is a metric space. Then we introduce normal coordinate systems at a given point. In these systems the computations are easier. We continue with the exponential mapping. Then we define some operators on the differential forms, and we conclude with the proofs of some theorems, using global notation. There are a lot of exercises and problems. Most of them extend the course itself, which is confined to the main topics.

Some Definitions

5.1. Definition. A $C^\infty$ Riemannian manifold of dimension $n$ is a pair $(M_n, g)$, where $M_n$ is a $C^\infty$ differentiable manifold and $g$ a Riemannian metric. A Riemannian metric is a twice-covariant tensor field $g$ (that is to say, a section of $T^*(M) \otimes T^*(M)$) such that, at each point $P \in M$, $g_P$ is a positive definite bilinear symmetric form:

$$g_P(X, Y) = g_P(Y, X)$$
$$g_P(X, X) > 0$$

for all $X, Y$, if $X \neq 0$.

In the sequel, $(M_n, g)$ will always be a connected $C^\infty$ Riemannian manifold endowed with the Riemannian connection (Definition 5.5). If the manifold is not connected, we study each of its connected components separately. If the manifold is $C^1$, we consider a $C^\infty$ atlas $C^1$-equivalent to the $C^1$ atlas.
5.2. Theorem. On a paracompact \( C^\infty \) differentiable manifold \( M \), there exists a \( C^\infty \) Riemannian metric \( g \).

Proof: Let \( (V_i, \varphi_i)_{i \in I} \) be an atlas of \( M \) and \( \{ \alpha_i \} \) a partition of unity subordinate to the covering \( \{ V_i \} \). This partition exists, since the manifold is paracompact (Theorem 1.12). \( \mathcal{E} = ((\mathcal{E}_{jk})) \) being the Euclidean metric on \( \mathbb{R}^n \) (in an orthonormal basis \( \mathcal{E}_{jk} = \delta^k_j \)), we consider the tensor field
\[
g = \sum_{i \in I} \alpha_i \varphi_i^* (\mathcal{E}).
\]

Let us verify that \( g \) is a Riemannian metric on \( M \). Indeed,
\[
g(X, Y) = \sum_{i \in I} \alpha_i \mathcal{E}(\varphi_i^* X, \varphi_i^* Y) = g(Y, X),
\]
according to the definition of the linear cotangent mapping and since \( \mathcal{E} \) is symmetric.

Moreover, \( g(X, X) = \sum_{i \in I} \alpha_i \mathcal{E}(\varphi_i^* X, \varphi_i^* X) > 0 \) if \( X \neq 0 \).

Indeed, at a point \( P \), at least one \( \alpha_i \) does not vanish. Let \( \alpha_{i_0}(P) \neq 0 \). Then \( X \neq 0 \) implies \( (\varphi_{i_0*})_P X \neq 0 \), since \( \varphi_{i_0*} \) is invertible. Thus
\[
g_P(X, X) \geq \alpha_{i_0}(P) \mathcal{E}(\varphi_{i_0*} X, \varphi_{i_0*} X) > 0.
\]

By using the Whitney Theorem 1.22, we can also define a Riemannian metric on \( M_n \): the imbedding metric. If \( h \) is an imbedding \( M \rightarrow \mathbb{R}^{2n+1} \), the imbedding metric \( g \) is \( g = h^* \mathcal{E} \). In a local chart \( (\Omega, \varphi) \) at a point \( P \in M \) let us compute the components of \( g \). If \( \{ x^i \} \) is the coordinate system on \( \Omega \) corresponding to \( (\Omega, \varphi) \), let \( \{ y^\alpha \} \) \( (\alpha = 1, 2, \ldots, 2n+1) \) be coordinates in \( (\mathbb{R}^{2n+1}, \mathcal{E}) \).

The imbedding \( h \) is defined on \( \Omega \) by \( 2n+1 C^\infty \)-functions
\[
y^\alpha(x^1, x^2, \ldots, x^n), \quad \alpha = 1, 2, \ldots, 2n+1.
\]

The components of the imbedding metric tensor are given on \( \Omega \) by
\[
g_{ij} = \mathcal{E}_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} = \sum_{\alpha=1}^{2n+1} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\alpha}{\partial x^j}.
\]

5.3. Definition. The length of a differentiable curve \( C : \mathbb{R} \supset [a, b] \rightarrow M_n \) of class \( C^1 \) is defined to be
\[
L(C) = \int_a^b \sqrt{g_C(t) \left( \frac{dC}{dt}, \frac{dC}{dt} \right)} dt.
\]

If a curve is piecewise differentiable, its length is the sum of the lengths of the pieces.
Some Definitions

We verify that the definition makes sense. It does not depend on the atlas or on the parameterization. If \( t = t(s) \) with \( dt/ds > 0 \), then \( \gamma(s) = C(t(s)) \) and \( C(t) \) have the same length.

A connected manifold is path connected (Proposition 1.3). For given \( P, Q \) in a \( C^\infty \) differentiable manifold \( M \), there is a \( C^\infty \) differentiable arc from \( P \) to \( Q \). Indeed, the continuous path \( \gamma \) from \( P \) to \( Q \) is compact, and is covered by a finite set of local charts \( (\Omega_i, \varphi_i), i = 1, 2, \cdots, m \), such that \( \varphi_i(\Omega_i) \) is a ball of \( \mathbb{R}^n \). Let \( P_i \) be in \( \Omega_i \cap \Omega_{i+1} \). Instead of the arc \( \gamma \) from \( P \) to \( P_i \) in \( \Omega_i \), we consider a \( C^\infty \) differentiable arc \( C_1 \) in \( \Omega_i \) from \( P \) to \( P_i \), for instance \( \varphi^{-1}_1([\varphi_1(P), \varphi_1(P_i)]) \), where \( \lfloor \cdot \rfloor \) means the segment in \( \mathbb{R}^n \). And so on, until the arc \( C_m \) from \( P_{m-1} \) to \( Q \). We get a piecewise \( C^\infty \) differentiable arc from \( P \) to \( Q \), the union of the arcs \( C_k \) (\( 1 \leq k \leq m \)). Then it is possible to smooth the corners at the points \( P_i \) to obtain a \( C^\infty \) differentiable arc from \( P \) to \( Q \).

5.4. Theorem. Set \( d(P, Q) = \inf \sum_{i=1}^{m} L(C_i) \), where the infimum is over all piecewise \( C^1 \)-differentiable paths \( C \) from \( P \) to \( Q \), \( C = \bigcup_{i=1}^{m} C_i \). Then \( (P, Q) \to d(P, Q) \) defines a distance on \( M_n \), and the topology determined by this distance is equivalent to the initial topology on \( M \).

Obviously \( d(P, Q) = d(Q, P) \) and \( d(P, Q) \leq d(P, R) + d(R, Q) \), since the inf of the length of the piecewise differentiable arcs from \( P \) to \( Q \) is less than or equal to the length of the piecewise differentiable arcs from \( P \) to \( Q \) through \( R \). It remains to prove that \( d(P, Q) = 0 \) implies \( P = Q \); we do it by contradiction.

Let \( (\Omega, \varphi) \) be a local chart at \( P \) with \( \varphi(P) = 0 \). If \( Q \neq P \), there is a ball \( B_r \subset \mathbb{R}^n \), of radius \( r \) with center \( O \), such that \( \overline{B_r} \subset \varphi(\Omega) \) and \( Q \notin \varphi^{-1}(\overline{B_r}) \).

Let us consider the map \( \Psi \) defined by
\[
\Sigma_{n-1} \times \varphi^{-1}(\overline{B_r}) \ni (\xi, x) \to g_x(\tilde{\xi}, \tilde{\xi}) \in \mathbb{R},
\]
where \( \Sigma_{n-1} \) is the unit sphere of dimension \( n - 1 \) and radius 1 in \( \mathbb{R}^n \), and \( \tilde{\xi} = \varphi^{-1}_x(\xi) \). Since \( \tilde{\xi} \neq 0 \), it follows that \( g_x(\tilde{\xi}, \tilde{\xi}) > 0 \).

Moreover, \( \Psi \) is continuous (in fact \( C^\infty \)) and \( K = \Sigma_{n-1} \times \varphi^{-1}(\overline{B_r}) \) is a compact set; thus \( \Psi \) achieves its minimum \( \lambda^2 \) and its maximum \( \mu^2 \) on \( K \) which satisfy \( 0 < \lambda \leq \mu < \infty \) and, for any \( \xi \in \mathbb{R}^n \) and \( x \in \varphi^{-1}(\overline{B_r}) \),
\[
\lambda^2 ||\xi||^2 \leq g_x(\tilde{\xi}, \tilde{\xi}) \leq \mu^2 ||\xi||^2,
\]
\( ||\xi|| \) being the Euclidean norm.

Let \( \Gamma \) be the connected component of \( P \) on the arc \( C \) from \( P \) to \( Q \) in \( \varphi^{-1}(\overline{B_r}) \). The end points of \( \Gamma \) are \( P \) and \( R \), \( C(a) = P, C(b) = R \). We have
\[
L(C) \geq L(\Gamma) = \int_a^b \sqrt{g_C(t)} \left| \frac{dC}{dt} \cdot \frac{dC}{dt} \right| dt \geq \lambda \int_a^b \left\| \frac{d(\varphi \circ C)}{dt} \right\| dt \geq \lambda r.
\]
Indeed,

\[ \varphi^* \frac{dC}{dt} = \frac{d(\varphi \circ C)}{dt}, \]

and the length of \( \varphi(\Gamma) \) in \( \mathbb{R}^n \) is at least \( r \). Thus \( d(P, Q) > 0 \) if \( P \neq Q \).

Setting \( S_r = \{ Q \in M \mid d(P, Q) \leq r \} \), we have, according to the proof above, \( S_{\lambda r} \subset \varphi^{-1}(B_r) \). Likewise we prove that \( \varphi^{-1}(B_r) \subset S_{\mu r} \). Indeed, we have

\[ d(P, R) \leq L(\Gamma) \leq \mu \int_a^b \left\| \frac{d(\varphi \circ C)}{dt} \right\| dt \]

for any arc \( \Gamma \) from \( P \) to \( R \). Picking \( \varphi \circ C \) to be the straight line from 0 to \( \varphi(R) \), we find that \( d(P, R) \leq \mu r \). Hence the topology defined by the metric is the initial topology.

### Riemannian Connection

#### 5.5. Definition. The Riemannian connection is the unique connection with vanishing torsion tensor, for which the covariant derivative of the metric tensor is zero (\( \nabla g = 0 \)).

Let us compute the expression of the Christoffel symbols in a local coordinate system. The computation gives a proof of the existence and uniqueness of the Riemannian connection.

The connection has no torsion; thus \( \Gamma^k_{ij} = \Gamma^k_{ji} \). Moreover, let us write that \( \nabla_k g_{ij} = \nabla_i g_{jk} = \nabla_j g_{ki} = 0 \):

\[
\begin{align*}
\partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il} &= 0, \\
\partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} &= 0, \\
\partial_j g_{ki} - \Gamma^l_{jk} g_{il} - \Gamma^l_{ji} g_{kl} &= 0.
\end{align*}
\]
Taking the sum of the last two equalities minus the first one, we find that
\[ \Gamma^l_{ij} = \frac{1}{2} [\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}] g^{kl}, \]
where the \( g^{kl} \) are, by definition, the components of the inverse matrix of the matrix \((g_{ij})\): \( g_{ij} g^{kj} = \delta^k_i \) (the Kronecker symbol).

5.6. With the metric tensor of components \( g_{ij} \), the indices go down by contraction. For example, if \( \xi^i \) \((1 \leq j \leq n)\) are the components of a vector field, then \( \alpha_i = g_{ij} \xi^j \) are the components of a 1-form. If \( R^k_{lij} \) are the components of the curvature tensor (see below), then \( R_{kl}^{ij} = g_{km} R^m_{lij} \) are the components of the 4-covariant curvature tensor.

For the tensor of components \( g^{ij} \), the indices go up by contraction. For example, if \( \alpha_i \) are the components of a 1-form, then \( \xi^j = g^{ij} \alpha_i \) are the components of a vector, and \( R^m_{lij} = g^{km} R_{kl}^{ij} \).

5.7. Definition. A normal coordinate system at \( P \in M_n \) is a local coordinate system for which the components of the metric tensor at \( P \) satisfy \( g_{ij}(P) = \delta^j_i \) and \( \partial_k g_{ij}(P) = 0 \) for all \( i, j, k \). (\( 4 g_{ii}(P) = 0 \) is equivalent to \( \Gamma^k_{ij}(P) = 0 \); see the proof above in 5.5.)

Let us prove that at each point \( P \), there exists a normal coordinate system. Let \( (\Omega, \varphi) \) be a local chart at \( P \) with \( \varphi(P) = 0 \), and \( \{x^i\} \) the corresponding coordinate system. First, by a linear transformation, we may choose a frame in \( \mathbb{R}^n \) so that \( \partial^j \delta_{ij}(P) = 0 \). Then consider the change of coordinates defined by
\[ x^k - y^k = -\frac{1}{2} \Gamma^k_{ij}(P) y^i y^j. \]
\( \{y^i\} \) is a coordinate system in a neighbourhood of \( P \) according to the inverse function theorem. Since the Jacobian matrix \((\partial x^k/\partial y^i))\) is invertible at \( P \), it is the identity matrix.

In the coordinate system \( \{y^i\} \), the components of the metric tensor are
\[ \tilde{g}_{ij}(y) = g_{kl}(x)[\delta^k_i - \Gamma^k_{im}(P)y^m][\delta^l_j - \Gamma^l_{jm}(P)y^m], \]
since \( \partial x^k/\partial y^j = \delta^k_j - \Gamma^k_{ij}(P)y^j \). \( x \) and \( y \) represent the same point \( Q \) in a neighbourhood of \( P \). The first order term in \( y^m \) of \( \tilde{g}_{ij}(y) - g_{ij}(x) \) is
\[ -[\Gamma^j_{im}(P) + \Gamma^i_{jm}(P)]y^m = -\left( \frac{\partial g_{ij}}{\partial x^m} \right)_P y^m = -\left( \frac{\partial g_{ij}}{\partial y^m} \right)_P y^m. \]
Hence \( \partial \tilde{g}_{ij}/\partial y^m \) is zero. The value of a normal coordinate system will be obvious when we consider, for instance, the proof of the Bianchi identities.
5.8. The curvature tensor.

Consider the 4-covariant tensor $R(X, Y, Z, T) = g[R(X, Y)T, Z]$; its components are $R_{ijkl} = g_{im}R_{jkl}^{m}$. It has the properties $R_{ijkl} = -R_{ijlk}$ (by definition), $R_{ijkl} = R_{klij}$, and

$$R_{ijkl} + R_{dijk} + R_{iklj} = 0,$$
$$\nabla_{m}R_{ijkl} + \nabla_{k}R_{ijlm} + \nabla_{l}R_{ijmk} = 0.$$

The last two equalities are the Bianchi identities. Let us prove them. Consider at $P$ a normal coordinate system. According to the expression of the components of the curvature tensor,

$$I_i (P) = (\partial_{jk}g_{i})P - (\partial_{lj}g_{ik})P,$$
$$R_{ijk}(P) = (\partial_{jk}g_{ki})P - (\partial_{lj}g_{ik})P,$$
$$R_{jki}(P) = (\partial_{jk}g_{ki})P - (\partial_{lj}g_{ik})P.$$

Taking the sum of these three equalities, we get zero. This is the first Bianchi identity (a tensor equality is proved when it is proved in some coordinate system). Differentiating the components of the curvature tensor at $P$, we find that

$$(\nabla_{m}R_{ijkl})P = (\partial_{km}g_{ij})P - (\partial_{lm}g_{ij})P,$$
$$(\nabla_{k}R_{ijlm})P = (\partial_{km}g_{ij})P - (\partial_{mk}g_{ij})P,$$
$$(\nabla_{l}R_{ijmk})P = (\partial_{lm}g_{kj})P - (\partial_{lk}g_{mj})P.$$

Taking the sum of these three equalities, we find that

$$\nabla_{m}R_{ijkl} + \nabla_{k}R_{ijlm} + \nabla_{l}R_{ijmk} = 0,$$

which is the second Bianchi identity. We put it into the form written above by multiplying it by $g_{ih}$. Since $\nabla g = 0$,

$$g_{ih}\nabla_{m}R_{ijkl} = \nabla_{m}(g_{ih}R_{ijkl}) = \nabla_{m}R_{hijkl}.$$

So we find that

$$\nabla_{m}R_{ijkl} + \nabla_{k}R_{ijlm} + \nabla_{l}R_{ijmk} = 0.$$

The equality remaining to be proved is $R_{ijkl} = R_{klij}$. In a normal coordinate system at $P$ we have

$$R_{klij}(P) = \frac{1}{2} [\partial_{jk}g_{il} + \partial_{il}g_{jk} - \partial_{ki}g_{jl} - \partial_{jl}g_{ik}]P.$$

In this expression we see that $R_{ijkl} = R_{klij}$. Let $\{\xi^{k}\}$ be the components of a vector field, and recall the following equality (see 4.15 with zero torsion):

$$\nabla_{i}\nabla_{j}\xi^{k} - \nabla_{j}\nabla_{i}\xi^{k} = R_{ij}^{l}\xi^{l}.$$

By definition, \( \sigma(X, Y) = R(X, Y, X, Y) \) is the sectional curvature of the 2-dimensional subspace of \( T(M) \) defined by the vectors \( X \) and \( Y \), \( X \) and \( Y \) being orthonormal (i.e., \( g(X, X) = 1 \), \( g(Y, Y) = 1 \), and \( g(X, Y) = 0 \)). Otherwise \( \sigma(X, Y) = R(X, Y, X, Y)/[g(X, X)g(Y, Y) - (g(X, Y))^2] \).

From the curvature tensor is it possible to obtain, by contraction, other tensors? We obtain only one nonzero tensor (or its negative); it is called the Ricci tensor. Its components are \( R_{ij} = R_{ikl}^k \). The Ricci tensor is symmetric:

\[
R_{ij} = g^{kl}R_{iklj} = g^{kl}R_{klij} = g^{lk}R_{klij} = R_{ji}
\]

according to the properties of the metric and curvature tensors. The Ricci curvature in the direction of the unit tangent vector \( X = \{e_i\} \) is \( R_{ij} \xi^i \xi^j \). The contraction of the Ricci tensor \( R = R_{ij}g^{ij} \) is called the scalar curvature.

Exponential Mapping

5.10. Let \((U, \varphi)\) be a local chart related to a normal coordinate system \( \{x^i\} \) at \( P \); we suppose \( \varphi(P) = 0 \). Also, let \( \bar{X} = \{\xi^i\} \neq \emptyset \) be a tangent vector of \( T_P(M) \). Let \( C^i(t) \) be the coordinates of the point \( C(t) \) belonging to the geodesic defined by the initial conditions \( C(0) = P \) and \( (dC/dt)_{t=0} = \bar{X} \). \( C(t) \) is defined for the values of \( t \) satisfying \( 0 \leq t < \beta \) (\( \beta \) given by the Cauchy theorem).

Then the quantity

\[
g_{ij}(C(t)) \frac{dC^i(t)}{dt} \frac{dC^j(t)}{dt}
\]

is constant along \( C \). Indeed, the covariant derivative along \( C \) of each of the three terms is zero: \( Dg = 0 \) and \( D_{dC/dt} \frac{dC}{dt} = 0 \). Hence \( s \), the parameter of arc length, is proportional to \( t \) (\( s = \|\bar{X}\|t \)):

\[
g_{ij}(C(t)) \frac{dC^i(t)}{dt} \frac{dC^j(t)}{dt} = g_{ij}(C(0)) \left( \frac{dC^i(t)}{dt} \right)_{t=0} \left( \frac{dC^j(t)}{dt} \right)_{t=0} = \|\bar{X}\|^2.
\]

The \( C^i(t) \) are \( C^\infty \) functions not only of \( t \), but also of the initial conditions. We may consider \( C^i(t, Q, \bar{X}) = C^i(t, x^1, x^2, \ldots, x^n, \xi^1, \ldots, \xi^n) \), where the \( \{x^i\} \) are the coordinates of \( Q \in U \) and \( \{\xi^i\} \) the components of \( \bar{X} \) in the basis \( \{\partial/\partial x^i\} \). According to the Cauchy Theorem 0.37, \( \beta \) may be chosen valid for initial conditions in an open set, for instance for \( Q \in \varphi^{-1}(B_r) \) and \( \|\bar{X}\| < \alpha \) (\( B_r \subset \varphi(U) \) being a ball centered at 0 of radius \( r > 0 \) and \( \alpha > 0 \)). As we will consider only geodesics \( C(t, P, \bar{X}) \) starting from \( P \), we will write \( C(t, \bar{X}) \) for \( C(t, P, \bar{X}) \).
Let us verify that \( C(t, \lambda \tilde{X}) = C(\lambda t, \tilde{X}) \) for all \( \lambda \), when one of the two numbers exists. Set \( \gamma(u) = C(\lambda u, \tilde{X}) \). Then

\[
\frac{d\gamma}{du} = \lambda \frac{dC}{dt} \quad \text{and} \quad \frac{d^2\gamma}{du^2} = \lambda^2 \frac{d^2C}{dt^2}.
\]

Thus, since \( C \) is a geodesic, \( \gamma \) satisfies the geodesic equation (4.10). \( \gamma \) is a geodesic starting from \( P \) such that \( (d\gamma/du)_{u=0} = \lambda \tilde{X} \); it is \( C(u, \lambda \tilde{X}) \). Hence in all cases, if \( \alpha \) is small enough, we may assume \( \beta > 1 \) without loss of generality.

5.11. Theorem. The exponential mapping \( \exp_P(X) \), defined by

\[
\mathbb{R}^n \ni \theta \ni X \rightarrow C(1, P, \tilde{X}) \in M_n,
\]

is a diffeomorphism of \( \theta \) (a neighbourhood of zero, where the mapping is defined) onto a neighbourhood \( \Omega \) of \( P \), \( \Omega = \exp_P \theta \). By definition, \( \exp_P(0) = P \), and the identification of \( \mathbb{R}^n \) with \( T_P(M) \) is made by means of \( \varphi_\ast \):

\[
\tilde{X} = (\varphi_\ast^{-1})_P X.
\]

Proof. We saw in 5.10 that \( \exp_P(X) \) is a \( C^\infty \) map from a neighbourhood of \( 0 \in \mathbb{R}^n \) into \( M_n \) (\( \beta \) may chosen greater than 1). At 0 the Jacobian of this map is 1; then, according to the inverse function theorem, the exponential mapping is locally a diffeomorphism (on \( \theta \)): \( \xi^1, \xi^2, \ldots, \xi^n \) can be expressed as functions of \( C^1, C^2, \ldots, C^n \). To compute the Jacobian matrix at 0, we compute the derivative in a given direction \( X \):

\[
\left( \frac{d}{d\lambda} \exp_P \lambda X \right)_{\lambda=0} = \left( \frac{d}{d\lambda} C(1, \lambda \tilde{X}) \right)_{\lambda=0} = \left( \frac{d}{d\lambda} C(\lambda, \tilde{X}) \right)_{\lambda=0} = \tilde{X} = (\varphi_\ast^{-1})_P X.
\]

Let \( B_0(r(P)) \) be the greatest ball with center 0 and radius \( r(P) \) in \( \theta \). \( r(P) \) is called the injectivity radius at \( P \), and \( r_1 = \inf_{P \in M} r(P) \) is called the injectivity radius of the manifold.

5.12. Corollary. There exists a neighbourhood \( \Omega \) of \( P \) such that every point \( Q \in \Omega \) can be joined to \( P \) by a unique geodesic entirely included in \( \Omega \). \( (\Omega, \exp_P^{-1}) \) is a local chart, and the corresponding coordinate system is called a normal geodesic coordinate system.

It remains to be proved that this coordinate system is normal at \( P \), since by 5.11 if we have \( Q \) (that is to say \( C^1, C^2, \ldots, C^n \)), then we have \( \xi^1, \xi^2, \ldots, \xi^n \) the components of \( \tilde{X} \) such that \( C(1, \tilde{X}) = Q \). In this new coordinate system, \( \{\xi^i\} \) are the coordinates of \( Q \in \Omega \).

Let \( C(t) = \{C^i(t)\} \) be the geodesic from \( P \) to \( Q \) lying in \( \Omega \). We verify that \( C^i(t) = t \xi^i \) for \( t \in [0, 1] \)—this comes from the equality \( C(t, \tilde{X}) = C(1, t\tilde{X}) \).
Exponential Mapping

The arc length $s = \|X\|t = \|X\|t$, so

$$g_{ij}(Q)\xi^i \xi^j = \sum_{i=1}^{n}(\xi^i)^2 = \|X\|^2,$$

and the length of the geodesic from $P$ to $Q$ is $\|X\|$.

Since $C(t)$ satisfies the geodesic equation (4.10), $d^2C^k/dt^2 = 0$ implies $\Gamma_{ij}^k[C(t)]\xi^i \xi^j = 0$. Letting $t \to 0$ gives $\Gamma_{ij}^k(P)\xi^i \xi^j = 0$ for all $\{\xi^i\}$. Then $\Gamma_{ij}^k = \Gamma_{ji}^k$ implies $\Gamma_{ij}^k(P) = 0$ for all $i, j, k$.

5.13. Proposition. Every geodesic starting from $P$ is perpendicular to $\Sigma_P(r)$, the image by $\exp_P$ of the sphere of center $0$ and radius $r$ in $\mathbb{R}^n$ when $r$ is small enough.

$\Sigma_P(r)$ is the subset of the points $Q \in \Omega$ satisfying $\sum_{i=1}^{n}(\xi^i)^2 = r^2$, where $\xi^i$ are the geodesic coordinates of $Q$.

Choose an orthonormal frame of $\mathbb{R}^n$ such that the geodesic coordinates of $Q \in \Sigma_P(r)$ are $\xi^1 = r$ and $\xi^2 = \xi^3 = \cdots = \xi^n = 0$. The desired result will be established if we prove that $g_{ii}(Q) = \delta_i^1$ for all $i$, because a vector $Y$ in $T_Q(M)$ tangent to $\Sigma_P(r)$ has a zero first component:

$$g_Y \left( \frac{dC}{dt}, Y \right) = \sum_{i=2}^{n} g_{ii}(Q) \frac{dC^i}{dt} Y^i = 0 \quad \text{if } g_{ii}(Q) = 0 \text{ for } i > 1.$$

Indeed, if $\gamma(u)$ is a differentiable curve in $\Sigma_P(r)$ through $Q$, then we have $\sum_{i=1}^{n}[\gamma^i(u)]^2 = r^2$ and $\sum_{i=1}^{n} \gamma^i(u)d\gamma^i(u)/du = 0$ by differentiation. Thus, since $\gamma^i(u) = 0$ at $Q$ for $i > 1$, that implies $d\gamma^1(u)/du = 0$ at $Q$.

We saw that $g_{ij}(\xi)\xi^i \xi^j = \sum_{i=1}^{n}(\xi^i)^2$, $\xi = \{\xi^i\}$. At $Q$ that gives $g_{ii}(Q) = 1$, and differentiation with respect to $\xi^k$ yields

$$\partial_k g_{ij}(Q)\xi^i \xi^j + 2g_{ik}(Q)\xi^i = 2\xi^k.$$

Hence, at $Q$, if $k \neq 1$,

$$r \partial_k g_{11}(r) + 2g_{1k}(r) = 0,$$
where \( g_{ij}(r) \) are the components of \( g \) at the point with coordinates \( \xi^1 = r, \xi^i = 0 \) for \( i > 1 \). Moreover, \( \Gamma^k_{ij}(r)\xi^j = 0 \) leads to \( 2\partial_r g_{1k}(r) = \partial_k g_{11}(r) \). Thus we get \( g_{1k}(r) + r\partial_r g_{1k}(r) = 0 \); that is, \( \frac{\partial}{\partial r} [rg_{1k}(r)] = 0 \) and \( rg_{1k}(r) \) is constant along the geodesic from \( P \) to \( Q \). It is zero at \( P \); hence \( g_{1k}(Q) = 0 \) for \( k \neq 1 \).

5.14. Theorem. A \( C^2 \) differentiable curve which minimizes the distance is a geodesic. If \( \|X\| \) is small enough, the distance from \( P \) to \( Q = \exp_P X \) is \( \|X\| \), which is the length of the unique geodesic from \( P \) to \( Q \) given by Corollary 5.12.

Proof. Let \( C \) be a minimizing \( C^2 \) differentiable curve from \( P \) to \( Q \) parameterized by the arc length \( t ([0, r] \ni t \to C(t) \in C) \), which lies in a chart \((\Omega, \varphi)\), and let \( \{x^i\} \) be the associated coordinate system. We will prove that it is a geodesic. Consider a family \( \{C_\lambda\} \) of \( C^2 \) differentiable curves from \( P \) to \( Q \) \((C_\lambda \subset \Omega \text{ for } \lambda \in \mathbb{R} \setminus [-\varepsilon, \varepsilon] \)) defined by \( C_\lambda(t) = \phi(t) + V(t) \) with \( C'_\lambda(0) = C_\lambda(0) = 0 \) for all \( i, t \to \xi^i(t) \) being \( C^\infty \) functions on \([0, r]\).

Since the quantity

\[
L(C_\lambda) = \int_0^r \sqrt{g_{\lambda}(t) \left( \frac{dC_\lambda}{dt}, \frac{dC_\lambda}{dt} \right)} dt
\]

attains its minimum at \( \lambda = 0 \), we have

\[
\left( \frac{dL(C_\lambda)}{d\lambda} \right)_{\lambda=0} = \frac{1}{2} \int_0^r \left[ \partial_k g_{ij}(C(t)) \xi^k \frac{dC^i}{dt} \frac{dC^j}{dt} + 2g_{ij}(C(t)) \frac{dC^i}{dt} \frac{dC^j}{dt} \right] dt = 0.
\]

As we have

\[
\frac{d}{dt} \left[ g_{ij}(C(t)) \frac{dC^j}{dt} \right] = g_{ij}(C(t)) \frac{d^2C^j}{dt^2} + \partial_k g_{ij}(C(t)) \frac{dC^k}{dt} \frac{dC^j}{dt},
\]

integration by parts of the second term gives

\[
\frac{1}{2} \int_0^r \xi^i \left\{ \partial_k g_{kj}(C(t)) - 2\partial_k g_{ij}(C(t)) \frac{dC^j}{dt} \frac{dC^k}{dt} - 2g_{ij}(C(t)) \frac{d^2C^j}{dt^2} \right\} dt = 0
\]

for all \( \xi \) vanishing at 0 and \( r \). Hence we obtain

\[
\frac{d^2C^i}{dt^2} + \Gamma^i_{jk}(C(t)) \frac{dC^j}{dt} \frac{dC^k}{dt} = 0.
\]

This is the Euler equation of the problem: "Minimize \( L(\Gamma) \) for all \( C^2 \) differentiable curves \( \Gamma \) from \( P \) to \( Q \)." It is also the geodesic equation (4.10). Thus \( C \) is a geodesic.
We choose $\varphi = \exp_P^{-1}$. If the geodesic coordinates of $Q$ are $\{\xi_i\}$, we saw that the equation of the geodesic from $P$ to $Q$ is $C^i(t) = t\xi^i$ for $1 \leq i \leq n$ ($t \in [0, 1]$), and its length is $\tau = \sqrt{\sum_{i=1}^n (\xi_i)^2}$.

Consider $\gamma : [0, 1] \ni t \rightarrow \gamma(t) \in M$, a differentiable curve from $P$ to $Q$ lying in $\Omega$ ($\gamma(0) = P$, $\gamma(1) = Q$). Let us prove that the length of $\gamma$ is greater than or equal to $\tau$. If $(\rho, \theta)$ are geodesic polar coordinates ($\theta \in S_{n-1}(1)$, the sphere of dimension $n-1$ and radius $1$), Proposition 5.13 implies that

$$ g_{ij}[\gamma(t)] \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \geq \left( \frac{d\rho(\gamma(t))}{dt} \right)^2 $$

with $\rho(\gamma(t)) = \sqrt{\sum_{i=1}^n (\gamma^i(t))^2}$. Thus

$$ L(\gamma) = \int_0^1 \sqrt{g_{ij}[\gamma(t)]} \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} dt \geq \int_0^1 \frac{d\rho(\gamma(t))}{dt} dt = \rho(\gamma(1)) = \tau. $$

Hence $d(P, Q)$, the distance from $P$ to $Q$, is equal to $\tau$, the length of the geodesic from $P$ to $Q$.

Some Operators on Differential Forms

5.15. Definition. Let $(M^n, g)$ be an oriented Riemannian manifold and $\mathcal{A}$ an atlas compatible with the orientation (all changes of charts have positive Jacobian). In the coordinate system $\{x^i\}$ corresponding to $(\Omega, \varphi) \in \mathcal{A}$, the differential $n$-form $\eta$ is by definition

$$ \eta = \sqrt{|g(x)|} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n, $$

where $|g(x)|$ is the determinant of the metric matrix $((g_{ij}))$. $\eta$ is a global differentiable $n$-form, called an oriented volume $n$-form, and it is nowhere zero.

Indeed, in another chart $(\theta, \Psi) \in \mathcal{A}$ such that $\theta \cap \Omega \neq \emptyset$, with coordinate system $\{y^\alpha\}$, consider the differentiable $n$-form $\tilde{\eta} = \sqrt{|g(y)|} dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n$.

On $\Omega \cap \theta$, let us consider $A^\alpha_i = \partial y^\alpha / \partial x^i$, $B^i_\beta = \partial x^i / \partial y^\beta$. The Jacobian matrix $A = ((A^\alpha_i)) \in GL(\mathbb{R}^n)^+$, the subgroup of $GL(\mathbb{R}^n)$ consisting of those matrices $A$ for which $\det A = |A| > 0$. We set $B = ((B^i_\beta))$. On $\Omega \cap \theta$ we have $g_{\alpha\beta}(y) = B^i_\alpha B^j_\beta g_{ij}(x)$, and hence $|g(y)| = |B|^2 |g(x)|$. Moreover,

$$ dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n = |A| dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n. $$

That yields $\tilde{\eta} = |B|^2 \sqrt{|g(x)|} |A| dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$. Since $|B| > 0$ and $|A||B| = 1$, we have $\tilde{\eta} = \eta$ on $\Omega \cap \theta$.

On an oriented Riemannian manifold $M_n$, we define the following operators on the differentiable forms.
5.16. Adjoint operator \(*\). The operator \(*\) associates to a \(p\)-form \(\alpha\) an \((n-p)\)-form \(*\alpha\), called the \emph{adjoint} of \(\alpha\), defined as follows. In a chart \((\Omega, \varphi)\), the components of \(*\alpha\) are

\[ (*\alpha)_{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_n} = \frac{1}{p!} \eta_{\lambda_1, \lambda_2, \ldots, \lambda_n} \alpha_{\mu_1 \mu_2, \ldots, \mu_p} g^{\lambda_{1, \mu_1}} \cdots g_{\lambda_p, \mu_p} \]

\[ = \frac{1}{p!} \eta_{\lambda_1, \lambda_2, \ldots, \lambda_n} \alpha_{\lambda_1, \lambda_2, \ldots, \lambda_p}. \]

See in 5.6 how the indices go up and down. Let us verify that

\[ *1 = \eta, \quad *\eta = 1, \quad **\alpha = (-1)^{p(n-p)} \alpha \]

and

\[ \alpha \wedge (*\beta) = (\alpha \beta)\eta, \]

where \(\beta\) is a \(p\)-form; here \((\alpha \beta)\) denotes the scalar product of \(\alpha\) and \(\beta\):

\[ (\alpha \beta) = \frac{1}{p!} \alpha_{\lambda_1, \lambda_2, \ldots, \lambda_p} \beta_{\lambda_1, \lambda_2, \ldots, \lambda_p}. \]

We will prove the formula above at a point \(P\). We choose a system of normal coordinates at \(P\). Thus \(\eta(P) = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n\).

Since \(*\) is linear, we can suppose without loss of generality that \(\alpha = \alpha_{1,2,\ldots,p} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^p\). \(*\alpha\) has only one component which is not zero, namely \((*\alpha(P))_{(p+1),(p+2),\ldots,n} = (\alpha(P))_{1,2,\ldots,p}\). Also,

\[ (**\alpha(P))_{1,2,\ldots,p} = \eta_{(p+1),(p+2),\ldots,n,1,2,\ldots,p} (*\alpha(P))_{(p+1),(p+2),\ldots,n} = (-1)^{p(n-p)} (\alpha(P))_{1,2,\ldots,p}. \]

Since the exterior product is bilinear, we can suppose that \(\alpha = \alpha_{1,2,\ldots,p} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^p\). Thus

\[ (\alpha \wedge (*\beta))_P = (\alpha(P))_{1,2,\ldots,p} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^p \]

\[ \wedge [((\beta(P))_{1,2,\ldots,p} dx^{p+1} \wedge dx^{p+2} \wedge \cdots \wedge dx^n] \]

\[ = (\alpha(P))_{1,2,\ldots,p} (\beta(P))_{1,2,\ldots,p} \eta = (\alpha \beta)_P \eta. \]

Note that the adjoint operator is an isomorphism between the spaces \(\Lambda^p(M)\) and \(\Lambda^{n-p}(M)\).

5.17. Co-differential \(\delta\). Let \(\alpha \in \Lambda^p(M)\). We define \(\delta \alpha\) by its components in a local chart as follows:

\[ (\delta \alpha)_{\lambda_1, \lambda_2, \ldots, \lambda_{p-1}} = -\nabla^\gamma \alpha_{\gamma, \lambda_1, \lambda_2, \ldots, \lambda_{p-1}}. \]

On functions, \(\delta\) vanishes: \(\delta f = 0\). For the definition of \(\delta\), the manifold need not be orientable.
Some Operators on Differential Forms

The differential $\left(p - 1\right)$-form is called the co-differential of $\alpha$ and has the following properties: on the $p$-forms

$$
\delta = \left(-1\right)^{p} \ast^{-1} d\ast, \quad \delta \delta = 0.
$$

The last result is a consequence of the first: $\delta \delta = - \ast^{-1} dd\ast = 0$.

To prove the first result at $P$, we choose a system of normal coordinates at $P$. Then

$$
\left[d\left(\ast \alpha\right)\right]_{\lambda_p, \ldots, \lambda_n} = \frac{1}{p! \left(n - p\right)!} \left[\varepsilon_{\mu_1, \mu_2, \ldots, \mu_p}^{\nu_1, \nu_2, \ldots, \nu_q} \nu_{\lambda_1, \ldots, \lambda_n} \eta^{\mu_1, \ldots, \mu_p}_{\mu_{p+1}, \ldots, \mu_n} \partial_{\mu_1} \alpha_{\mu_2, \ldots, \mu_p}\right]
$$

In $\varepsilon_{\mu_1, \mu_2, \ldots, \mu_p}^{\nu_1, \nu_2, \ldots, \nu_q}$, $\nu_1, \ldots, \nu_q$ is a permutation of $\mu_1, \mu_2, \ldots, \mu_q$; according to the sign of the permutation $\varepsilon_{\mu_1, \ldots, \mu_p}^{\nu_1, \ldots, \nu_q} = \pm 1$. We saw that $\ast^{-1} = \left(-1\right)^{\left(p-1\right)\left(n-p+1\right)} \ast$ on an $(n - p + 1)$-form; hence

$$
\left(\delta \alpha\right)_{\nu_1, \ldots, \nu_{p-1}} = \left(-1\right)^{p+\left(p-1\right)\left(n-p+1\right)} \frac{1}{p! \left(n - p\right)! \left(n - p + 1\right)!} \varepsilon_{\lambda_1, \ldots, \lambda_n}^{\mu_1, \ldots, \mu_p} \varepsilon_{\mu_1, \ldots, \mu_p}^{\nu_1, \ldots, \nu_{p-1}} \eta^{\mu_1, \ldots, \mu_{p+1}, \ldots, \mu_n}_{\nu_1, \ldots, \nu_{p-1}} \partial_{\mu_{p+1}} \alpha_{\mu_{p+2}, \ldots, \mu_p}.
$$

Since

$$
\eta^{\lambda_1, \ldots, \lambda_n}_{\nu_1, \ldots, \nu_{p-1}} \varepsilon_{\mu_1, \ldots, \mu_p}^{\nu_1, \ldots, \nu_{p-1}} \eta^{\mu_1, \ldots, \mu_p}_{\mu_{p+1}, \ldots, \mu_n} = \eta^{\mu_1, \ldots, \mu_{p+1}, \ldots, \mu_n}_{\nu_1, \ldots, \nu_{p-1}} \varepsilon_{\lambda_1, \ldots, \lambda_n}^{\mu_1, \ldots, \mu_p}
$$

and

$$
\eta^{\mu_1, \ldots, \mu_{p+1}, \ldots, \mu_n}_{\nu_1, \ldots, \nu_{p-1}} \eta^{\mu_1, \ldots, \mu_p}_{\nu_1, \ldots, \nu_{p-1}} = \left(-1\right)^{\left(n-p\right)\left(p-1\right)} \varepsilon_{\mu_1, \ldots, \mu_{p+1}, \ldots, \mu_n}^{\nu_1, \ldots, \nu_{p-1}} \nu_{\lambda_1, \ldots, \lambda_n}
$$

it follows that

$$
\left(\delta \alpha\right)_{\nu_1, \ldots, \nu_{p-1}} = \frac{1}{p!} \varepsilon_{\nu_1, \ldots, \nu_{p-1}}^{\mu_1, \ldots, \mu_p} \nabla^{\mu} \alpha_{\mu_{p+1}, \ldots, \mu_p} = - \nabla^{\mu} \alpha_{\mu_{p+1}, \ldots, \mu_p}.
$$

5.18. The Laplacian operator $\Delta$ is defined by

$$
\Delta = d\delta + \delta d.
$$

If $\alpha$ is a differential $p$-form, then $\Delta \alpha$ is a differential $p$-form and the Laplacian commutes with the adjoint operator: $\Delta \ast = \ast \Delta$.

Indeed, on differential $p$-forms

$$
\ast \delta d = \left(-1\right)^{p+1} d \ast d = \left(-1\right)^{p+1+p\left(n-p\right)} d \ast d \ast = \left(-1\right)^{1+p\left(n-p+1\right)} d\left[\left(-1\right)^{\left(p+1\right)\left(n-p-1\right)} \ast^{-1} d\ast\right] \ast = \left(-1\right)^{\left(n-p\right)} d\left(\ast^{-1} d\ast\right) \ast = d\delta \ast.
$$

For a function $f$,

$$
\Delta f = \delta df = - \nabla^{\nu} \nabla_{\nu} f,
$$

$\alpha$ is said to be closed if $d\alpha = 0$, co-closed if $\delta \alpha = 0$, and harmonic if $\Delta \alpha = 0$. $\alpha$ is said to be exact or homologous to zero if there exists a differential form $\beta$ such that $\alpha = d\beta$. $\alpha$ is said to be co-exact or cohomologous to zero if
there exists a differential form \( \gamma \) such that \( \alpha = \delta \gamma \). Two differential \( p \)-forms are homologous if their difference is exact.

5.19. Global scalar product. On a compact orientable Riemannian manifold, the global scalar product \( \langle \alpha, \beta \rangle \) of two differential \( p \)-forms \( \alpha \) and \( \beta \) is defined as follows:

\[
\langle \alpha, \beta \rangle = \int_M (\alpha, \beta) \eta.
\]

Recall that \( \langle \alpha, \beta \rangle = \frac{1}{p!} \alpha_{\lambda_1, \lambda_2, \ldots, \lambda_p} \beta^{\lambda_1, \lambda_2, \ldots, \lambda_p} \). The name of the operator \( \delta \) comes from the formula

\[
\langle d\alpha, \gamma \rangle = \langle \alpha, \delta \gamma \rangle
\]

for all differential \( (p + 1) \)-forms \( \gamma \). \( \delta \) is the adjoint operator of \( d \) for the global scalar product. Let us prove this formula. According to 5.16

\[
\int (\alpha, \delta \gamma) \eta = \int \alpha \wedge \delta \gamma = (-1)^{p+1} \int \alpha \wedge d \star \gamma.
\]

But \( d(\alpha \wedge \gamma) = d\alpha \wedge \gamma + (-1)^p \alpha \wedge d \star \gamma \), and so

\[
\langle \alpha, \delta \gamma \rangle = \int d\alpha \wedge \gamma - \int d(\alpha \wedge \gamma) = \langle d\alpha, \gamma \rangle
\]

according to Stokes’ formula. Using this result yields

\[
\langle \Delta \alpha, \beta \rangle = \langle \delta \alpha, \delta \beta \rangle + \langle d\alpha, d\beta \rangle.
\]

\( \Delta \) is an elliptic self-adjoint differential operator. If \( f \in C^2(M) \), then

\[
\langle \Delta f, f \rangle = \langle df, df \rangle = \int_M \nabla^\nu f \nabla_\nu f \eta.
\]

5.20. Proposition. On a compact oriented Riemannian manifold \( M \) any harmonic form is closed and co-closed. A harmonic function is constant.

This follows from the equality above with \( \beta = \alpha \):

\[
\langle \Delta \alpha, \alpha \rangle = \langle \delta \alpha, \delta \alpha \rangle + \langle d\alpha, d\alpha \rangle.
\]

\( \Delta \alpha = 0 \) implies \( \delta \alpha = 0 \) and \( d\alpha = 0 \). If \( \delta \alpha = 0 \) and \( d\alpha = 0 \), then \( \Delta \alpha = 0 \) by definition. A harmonic function satisfies \( df = 0 \); thus the function \( f \) is constant.

5.21. Remark. If \( M \) is not orientable, the statement remains true. We consider a two-sheeted covering manifold \( \tilde{M} \) of \( M \) such that \( \tilde{M} \) is orientable \( (\pi : \tilde{V} \to V) \), and \( \tilde{g} = \pi^*g \) (Theorem 2.30). We work on \( \tilde{\alpha} = \pi^*\alpha \), which is harmonic. We have \( d\tilde{\alpha} = 0 \) and \( \delta \tilde{\alpha} = 0 \); thus \( d\alpha = 0 \) and \( \delta \alpha = 0 \).
5.22. Hodge Decomposition Theorem. Let $M_n$ be a compact and orientable Riemannian manifold. A differential $p$-form $\alpha$ may be decomposed into the sum of three differential $p$-forms:

$$d\alpha = d\lambda + \delta \mu + H\alpha,$$

where $H\alpha$ is a harmonic differential $p$-form, $d\lambda$ is exact and $\delta \mu$ co-exact. The decomposition is unique.

For the proof, see de Rham [11].

Uniqueness comes from the orthogonality of the three spaces for the global scalar product:

$$\langle \alpha, \alpha \rangle = \langle d\lambda, d\lambda \rangle + \langle \delta \mu, \delta \mu \rangle + \langle H\alpha, H\alpha \rangle.$$

The dimension $b_p$ of the space $H_p(M_n)$ of harmonic $p$-forms is called the $p^{th}$ Betti number of $M$. It is finite.

Since $\Delta \ast = \ast \Delta$ (see 5.18), $\ast$ defines an isomorphism between the spaces $H_p(M_n)$ and $H_{n-p}(M_n)$. Hence $b_p(M_n) = b_{n-p}(M_n)$. The number

$$\chi(M_n) = \sum_{p=0}^{n} (-1)^p b_p(M_n)$$

is called the Euler-Poincaré characteristic. Clearly, $b_0(M_n) = b_n(M_n) = 1$ (Proposition 5.20). $H_0(M_n)$ is the space of constant functions and $H_n(M_n)$ is the space of differential $n$-forms proportional to the oriented volume $n$-form $r!$. Indeed, $r!$ is harmonic since $\eta = \ast 1$; or we can see directly that $d\eta = 0$ and $\delta \eta = 0$. There is no nonzero $(n+1)$-form, and, since $\nabla g = 0$,

$$(\delta \eta)_{\lambda_1, \lambda_2, \ldots, \lambda_{n-1}} = -\frac{1}{n!} \epsilon^{1,2,\ldots,n}_{\nu, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}} \nabla^\nu \sqrt{|g|} = 0.$$

For a connected manifold, $b_n = 1$ or $b_n = 0$ according to whether it is orientable or not. In general, $b_0$ is the number of connected components.

Spectrum of a Manifold

5.23. The Lebesgue Integral. Let $(M_n, g)$ be a Riemannian manifold and $(\Omega, \varphi)$ a local chart, with $\{x^i\}$ the associated coordinate system. For a continuous function $f$ on $M_n$ with compact support lying in $\Omega$, we set

$$\int_M f dV = \int_{\varphi(\Omega)} (\sqrt{|g|} f) \circ \varphi^{-1} dx^1 dx^2 \cdots dx^n.$$

For a continuous function $f$ on $M_n$ with compact support, we set

$$\int_M f dV = \sum_{i \in I} \int_M \alpha_i f dV,$$
where \( \{ \alpha_i \}_{i \in I} \) is a partition of unity subordinate to the covering \( \{ \Omega_i \}_{i \in I} \), \((\Omega_i, \varphi_i)_{i \in I}\) being an atlas.

In the sum, only a finite number of terms are nonzero, since \( K \cap \text{supp} \alpha_i \) is empty except for a finite set of indices \( i \) if \( K \) is compact.

We have to prove that this definition makes sense. It depends neither on the local chart nor on the partition of unity. Indeed, let \(( \theta, \psi)\) be another chart, \(\{ y^a \}\) the associated coordinate system. Suppose \( \text{supp} f \subset \Omega \cap \theta \); set as usual \( \partial y^a / \partial x^i = A_i^a \) and \( \partial x^j / \partial y^b = B_j^b \) (see 5.15). Then

\[
\int_{\varphi(\Omega \cap \theta)} (\sqrt{|g(x)|} f) \circ \varphi^{-1} dx^1 dx^2 \ldots dx^n = \int_{\psi(\Omega \cap \theta)} (\sqrt{|g(y)|} f) \circ \psi^{-1} dy^1 dy^2 \ldots dy^n,
\]

since \(|g(y)| = |B|^2 |g(x)|\) and \(dy^1 dy^2 \ldots dy^n = ||A||dx^1 dx^2 \ldots dx^n\).

Consider another atlas \(\{ \theta_j, \psi_j \}_{j \in J}\) and a partition of unity \(\{ \beta_j \}_{j \in J}\) subordinate to the covering \(\{ \theta_j \}_{j \in J}\). We have

\[
\sum_{i \in I} \int_M \alpha_i f dV = \sum_{i \in I} \sum_{j \in J} \int_M (\alpha_i \beta_j) dV = \sum_{j \in J} \int_M \beta_j f dV,
\]

since the sums are finite.

Hence \( f \to \int_M f dV \) defines a positive Radon measure, and the theory of the Lebesgue integral can be applied. We call \(dV = \sqrt{|g(x)|} dx^1 dx^2 \ldots dx^n\) the Riemannian volume element.

5.24. Proposition. Let \(M_n\) be a compact Riemannian manifold, and \(\omega\) a differential 1-form. Then \(\int_M \delta \omega dV = 0\). In particular, if \(f \in C^2(M)\), then \(\int_M \Delta f dV = 0\).

If \(M\) is orientable, by choosing the correct orientation we have

\[
\int_M \delta \omega dV = \int_M \delta \omega \eta = \langle \delta \omega, 1 \rangle = \langle \omega, d1 \rangle = 0.
\]

If \(M\) is nonorientable, we consider an orientable two-sheeted Riemannian covering manifold \(\tilde{M}\) of \(M\) (Theorem 2.30). Let \(\pi\) be the covering map \(\tilde{M} \to M\), and let \(\tilde{g} = \pi^* g\) and \(\tilde{\omega} = \pi^* \omega\).

Since \(\tilde{M}\) is orientable, \(\int_{\tilde{M}} \tilde{\delta} \omega d\tilde{V} = 0\). Moreover,

\[
\int_{\tilde{M}} \tilde{\delta} \omega d\tilde{V} = 2 \int_{M} \delta \omega dV.
\]
Indeed, let \( \{\alpha_i\}_{1<i<m} \) be a partition of unity subordinate to the covering \( \{\Omega_P\}_{P \in M}, \Omega_P \) being a neighbourhood of \( P \) such that \( \pi^{-1}(\Omega_P) \) is diffeomorphic to \( \Omega_P \times F \), where \( F \) consists of two points. We have

\[
\int_M \delta \tilde{\alpha}_i \tilde{\omega} d\tilde{V} = 2 \int_M \delta(\alpha_i \omega) dV,
\]
whence the result follows. If \( f \in C^2(M) \), then \( \int_M \delta(df) dV = 0 \).

**5.25. Theorem.** Let \( M \) be a compact Riemannian manifold with strictly positive Ricci curvature. Then \( b_1(M) \), the first Betti number of \( M \), is zero.

Let \( \alpha \) be a harmonic 1-form. Then, by Proposition 5.20 and Remark 5.21, \( d\alpha = 0 \) and \( \delta \alpha = 0 \). That is to say, we have \( \partial_i \alpha_j = \partial_j \alpha_i \) and \( \nabla^i \alpha_i = 0 \) in a local coordinate system where \( \alpha = \alpha_i dx^i \). But \( \partial_i \alpha_j = \partial_j \alpha_i \) implies \( \nabla_i \alpha_j = \partial_i \alpha_j - \Gamma^k_{ij} \alpha_k = \partial_j \alpha_i - \Gamma^k_{ji} \alpha_k = \nabla_j \alpha_i \).

Contracting the equality at the end of 5.8 (\( i = k \)) with \( \xi^k = \alpha^k \) gives

\[
(*): R_{ij} \alpha^i = \nabla_i (\nabla_j \alpha^i) - \nabla_j (\nabla_i \alpha^i).
\]
Then multiplying by \( \alpha^j \) and integrating over \( M \) lead to

\[
\int_M R_{ij} \alpha^i \alpha^j dV = \int_M \nabla_i [\alpha^j (\nabla_j \alpha^i)] dV - \int_M (\nabla_i \alpha_j) (\nabla^j \alpha^i) dV,
\]

since \( \nabla^i \alpha_i = 0 \). Moreover, according to Proposition 5.24,

\[
\int_M \nabla_i [\alpha^j (\nabla_j \alpha^i)] dV = 0.
\]

Hence, if \( \alpha \) does not vanish everywhere, the left hand side will be strictly positive, while the right hand side is \( \leq 0 \), since \( \nabla_i \alpha_j = \nabla_j \alpha_i \).

**5.26.** Let \( M \) be a compact Riemannian manifold. The spectrum of \( M \) is \( \text{SS}(M) = \{ \lambda \in \mathbb{R} | \text{ there exists } f \in C^2(M), f \neq 0, \text{ satisfying } \Delta f = \lambda f \} \).

\( \lambda \) is called an eigenvalue of the Laplacian, and \( f \) an eigenfunction. If \( \lambda \in \text{SS}(M) \), then \( \lambda \geq 0 \), because

\[
\lambda \int_M f^2 dV = \int_M f \Delta f dV = \int_M \nabla \nu f \nabla^\nu f dV \geq 0.
\]

The eigenvalues of the Laplacian form an infinite sequence \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \) going to \( +\infty \). And for each eigenvalue \( \lambda_i \), the set of corresponding eigenfunctions forms a vector space of finite dimension (Fredholm's theorem). For \( \lambda_0 \) the vector space is one-dimensional.

**5.27. Lichnerowicz's theorem.** Let \( M \) be a compact Riemannian manifold. If its Ricci curvature is greater than or equal to \( k > 0 \), then \( \lambda_1 \geq nk/(n-1) \).
Let \( f \) be an eigenfunction: \( \Delta f = \lambda f \) with \( \lambda > 0 \). Multiplying formula (*) used in 5.25, with \( \alpha = df \), by \( \nabla^j f \), and integrating over \( M \) lead to
\[
\int_M R_{ij} \nabla^i f \nabla^j f \, dV = \int_M \nabla^i f \nabla^i (\nabla_j \nabla_i f) \, dV + \lambda \int_M \nabla^j f \nabla_j f \, dV.
\]
According to the hypothesis, the 2-tensor \( R_{ij} - kg_{ij} \) is non-negative. Thus \( R_{ij} \nabla^i f \nabla^j f \geq k \nabla^i f \nabla_i f \). Integrating the second integral by parts gives
\[
\int_M \nabla^i f \nabla^i (\nabla_j \nabla_i f) \, dV = \int_M \nabla^i [\nabla^j f \nabla_j \nabla_i f] \, dV - \int_M \nabla^i \nabla^j f \nabla_j \nabla_i f \, dV
= - \int_M \nabla^i \nabla^j f \nabla_i \nabla_j f \, dV,
\]
thanks to Proposition 5.24 and the equality \( \nabla_i \nabla_j f = \nabla_j \nabla_i f \). At once we get \( \lambda \geq k \). But we have more. Since
\[
(\nabla_j \nabla f + \Delta g_{ij}/n)(\nabla^i \nabla^j f + \Delta g^{ij}/n) \geq 0,
\]
it follows that \( \nabla_i \nabla_j f \nabla^i \nabla^j f \geq (\Delta f)^2/n \), because \( g^{ij} g_{ij} = n \). Hence
\[
\lambda \int_M \nabla_j f \nabla^j f \, dV - \frac{\lambda}{n} \int_M f \Delta f \, dV \geq k \int_M \nabla^i f \nabla_i f \, dV.
\]
Since
\[
\int_M f \Delta f \, dV = - \int_M \nabla^i (f \nabla_i f) \, dV + \int_M \nabla^i f \nabla_i f \, dV
= \int_M \nabla^i f \nabla_i f \, dV \neq 0,
\]
the result follows.

5.28. Remark. The inequality of Theorem 5.27 is the best possible. Indeed, for the sphere \( S_n(1) \) of dimension \( n \) and radius 1, the sectional curvature is 1. Thus the Ricci curvature is \( n - 1 \). If \( k = n - 1 \), we get \( \lambda_1 \geq n \), and we will see that \( n \) is an eigenvalue for the Laplacian on \( S_n(1) \) (see Exercise 5.36 or Problem 5.38).

Exercises and Problems

5.29. Exercise. An Einstein metric is a metric for which the Ricci tensor and the metric tensor are proportional at each point.

If the metric is Einstein, prove that the scalar curvature \( R \) of the Riemannian manifold is constant when the dimension \( n > 2 \).

5.30. Exercise. Let \( (M_n, g) \) be a compact, oriented Riemannian manifold of dimension \( n > 1 \). When \( X \) is a \( C^\infty \) vector field on \( M_n \), we associate to \( X \) the 1-form \( \alpha \) such that \( \alpha(Y) = g(X, Y) \) for all vectors \( Y \in T(M) \).

a) For an oriented volume \( n \)-form \( \eta \), express \( \mathcal{L}_X \eta \) in terms of \( \delta \alpha \) and \( \eta \).
b) In a local coordinate system, express the components of $\Delta \alpha$ by means of the components $\alpha_i$ of $\alpha$, $\nabla^j \nabla_j \alpha_i$, and the components of the Ricci tensor.

c) Show that a necessary and sufficient condition for $\mathcal{L}_X g = 0$ is that $\nabla_i \alpha_j + \nabla_j \alpha_i = 0$ for all $(i,j)$.

d) Prove that $\mathcal{L}_X g = 0$ if and only if $d\delta \alpha = 0$ and $\Delta \alpha = 2R_{ij}X^i dx^j$, where the $X^j$ are the components of $X$.

e) Let $G$ be the one parameter local group of local diffeomorphisms $\varphi_t(X)$ associated to $X$.

Verify that $d(\varphi_t^*g)/dt = \varphi_t^* \mathcal{L}_X g$.

Deduce from this equality that $\varphi_t^* g = g$ if $\mathcal{L}_X g = 0$.

f) Let $\omega$ be a harmonic $p$-form. Show that $\mathcal{L}_X \omega = 0$ if $\mathcal{L}_X g = 0$.

Hint. If $\mathcal{L}_X g = 0$, then $\mathcal{L}_X \delta = \delta \mathcal{L}_X$.

5.31. Problem. Let $(M, g)$ be a $C^\infty$ Riemannian manifold with zero curvature ($R_{ijkl} = 0$).

Prove that there exists an atlas whose local charts $(\Omega, \varphi)$ are such that $\varphi^*\mathcal{E} = g$, where $\mathcal{E}$ is the Euclidean metric on $\mathbb{R}^n$.

Hint. Let $(\theta, \psi)$ be a local chart at $P$ and $\{x^i\}$ the corresponding coordinate system. We have to exhibit a local chart $(\Omega, \varphi)$ at $P$ with coordinate system $\{y^a\}$ such that $\varphi^*\mathcal{E} = g$. On $\theta$ set $\omega^k_j = \Gamma^k_{ij} dx^i$, and in a neighbourhood of $P$ set $A^a_i = \partial y^a_i / \partial x^i$.

a) Express $d\omega^k_j$ in terms of $\omega^l_i$.

b) Which system of differential equations must the $A^a_i$ satisfy?

c) Represent this system as a Pfaff system of order $n^2$ on a space of dimension $p$ ($p$ to be specified) and use the Frobenius theorem.

d) Solve the problem.

5.32. Problem. Let $M_n$ be a $C^\infty$ submanifold of $\mathbb{R}^p$ endowed with the Euclidean metric $\mathcal{E}$, and let $\tilde{\mathcal{D}}$ be the Riemannian connection on $(\mathbb{R}^p, \mathcal{E})$. Denote by $\Phi$ the inclusion $M_n \subset \mathbb{R}^p$, and consider on $M_n$ the imbedding metric $g = \Phi^*\mathcal{E}$. Let $\{x^i\}$ be the coordinate system in a neighbourhood of $Q \in M_n$ associated with the chart $(\Omega, \varphi)$, and $\{y^a\}$ the coordinates on $\mathbb{R}^p$.

a) Recall the expression of the components $g_{ij}(x)$ of $g$ as functions of the coordinates $y^a(x)$ of the point $\Phi(x)$, $x \in \Omega$.

By $\Phi_*$ we identify a tangent vector of $M_n$ to a tangent vector of $\mathbb{R}^p$ ($T(M_n) \subset T(\mathbb{R}^p)$). Let $Y \subset T(\mathbb{R}^p)$ be a vector field defined on $M_n$, and let $X \in T_Q(M_n)$.

Show that $\tilde{\mathcal{D}}_X Y$ is well defined.
b) Let \( \pi_Q \) be the orthogonal projection of vectors of \( T_Q(\mathbb{R}^p) \) onto \( T_Q(M_n) \). If \( Y \in T(M_n) \) is a vector field on \( M_n \), prove that \( D_X Y = \pi_Q \tilde{D}_X Y \) defines a connection on \( M_n \) which is the Riemannian connection of \((M_n, g)\).

c) For two vector fields \( X \) and \( Y \) on \( M_n \), we set \( H(X, Y) = \tilde{D}_X Y - D_X Y \). Verify that \( H(X, Y) = H(Y, X) \).

d) If \( p = n + 1 \) and if \( M_n \) is orientable, show that there exists on \( \Phi(M_n) \) a differentiable vector field \( N \subset T(\mathbb{R}^{n+1}) \) of unit Euclidean norm, orthogonal to \( T_Q(M_n) \) at each point \( Q \in M_n \). In this case verify that we can write \( H(X, Y) = h(X, Y)N \), \( h \) being a symmetric bilinear form satisfying \( h(X, Y) = -\mathcal{E}(Y, \tilde{D}_X N) \).

5.33. Problem. Let \((M_n, g)\) be a connected and complete \( C^\infty \) Riemannian manifold of sectional curvature greater than or equal to \( k^2 > 0 \). We consider a closed geodesic \( \gamma \); that is, \( \gamma \) is a differentiable map of the circle into \( M \). We assume the following result: any pair \((P, Q)\) of points of \( M \) can be joined by a geodesic arc whose length is equal to \( d(P, Q) \).

a) Let \( x \notin \gamma \) be a point of \( M \), and set \( d(x, \gamma) = \inf_{y \in \gamma} d(x, y) \).

Show that there is a point \( y_0 \) of \( \gamma \) with \( y_0 = \gamma(u_0) \), such that \( d(x, y_0) = d(x, \gamma) \). We set \( d(x, \gamma) = a \).

b) Let us consider a geodesic \( C \) from \( x \) to \( y_0 \) of length \( L(C) = a \). Let \([0, a] \ni t \to C(t) \in M \) with \( C(0) = x \) and \( C(a) = y_0 \). Do we have \( d(z, y_0) = d(z, \gamma) \) when \( z \in C \)?

\( \Sigma_b \) will be the set of points \( Q \in M \) of the form \( Q = C(1, z, \tilde{X}) \) (see 5.10 for the notations) with \( g_z(\tilde{X}, \tilde{X}) = b^2 \). If \( b \) is small enough, prove that \( d(z, Q) = b \) when \( Q \in \Sigma_b \).

c) Show that \( Y = (d\gamma/du)_{u=u_0} \) is orthogonal to \( (dC/dt)_{t=a} \).

\textit{Hint.} Consider a local chart \((\Omega, \exp_z^{-1})\) at \( z \in C \) close enough to \( y_0 \) so that \( y_0 \in \Omega \).

d) Let \( e(t) \) be the parallel vector field along \( C \) such that \( e(a) = Y \). Verify that \( g_{C(t)}(e(t), \frac{dC}{dt}) = 0 \) and that \( g_{C(t)}(e(t), e(t)) = g_{y_0}(Y, Y) \).

e) We consider the family of curves \( C_\lambda \) defined for \( \lambda \in ]-\epsilon, \epsilon[ \) (\( \epsilon > 0 \) small) by

\[ [0, a] \ni t \to C_\lambda(t) = C \left( \lambda \sin \frac{\pi t}{2a}, C(t), e(t) \right). \]

Verify that \( C_\lambda \) is a differentiable curve with endpoints \( x \) and a point \( \gamma(u_1) \) of \( \gamma \). What is the value of \( u_1 \)?

f) We will admit the existence of a neighbourhood \( \theta \) of \( C \) and of a local chart \((\theta, \varphi)\) such that the corresponding coordinate system is normal
Exercises and Problems

at any point of $C$. We set $f(\lambda) = L(C_\lambda)$. Prove that

$$f'(0) = \int_0^a g_{C(t)}(v(t), \frac{dC}{dt}) dt$$

for some vector field $v$. Verify that $f'(0) = 0$.

g) From the fact that $f(\lambda)$ is minimum at $\lambda = 0$, deduce that $d(x, \gamma) \leq \pi/2k$.

5.34. Exercise. Let $(M_n, g)$ be a complete $C^\infty$ Riemannian manifold and $C$ a geodesic through $P$: $[0, \alpha] \ni t \to C(t)$ with $C(0) = P$.

a) For an increasing sequence $\{t_i\} \subset \mathbb{R}$ converging to $\alpha$, show that the sequence $C(t_i)$ converges to a point $Q \in M$.

b) Consider the map $\exp_Q$. It is a diffeomorphism of a ball $\theta = B_0(\tau)$ in $\mathbb{R}^n$ onto a neighbourhood of $Q$ in $M$. Verify that there exists an $i_0$ such that $C(t_i) \in \Omega = \exp_Q(\theta)$ for $i \geq i_0$, and prove that the geodesic $C$ from $P$ to $Q$ can be extended beyond $Q$ (for $t \in [\alpha, \alpha + \varepsilon]$ with $\varepsilon > 0$).

c) Deduce from b) that on a complete Riemannian manifold all geodesics are infinitely extendable.

5.35. Exercise. Let $(M_n, g)$ be a compact and oriented $C^\infty$ Riemannian manifold.

a) Let $\gamma \in \Lambda^p(M)$. Show that $\Delta \alpha = \gamma$ has a solution $\alpha \in \Lambda^p(M)$ if and only if $\gamma \in A_p$, the set of the differential $p$-forms which are orthogonal to $H_p$ for the global scalar product. Here $H_p$ is the set of harmonic differential $p$-forms.

b) Exhibit a map $G : A_p \to A_p$ such that $\Delta G = G = Identity$ on $A_p$.

G is defined for each $p (0 \leq p \leq n)$.

c) Verify that $dG = Gd$, $\delta G = G\delta$ and $G^* = *G$.

d) For two closed differential forms $\gamma \in \Lambda^p(M)$ and $\phi \in \Lambda^{n-p}(M)$, we denote by $\tilde{\gamma}$ (respectively $\tilde{\phi}$) the set of differential $p$-forms homologous to $\gamma$ (respectively $\phi$). When $\gamma' \in \tilde{\gamma}$ and $\phi' \in \tilde{\phi}$, show that $\int \gamma' \wedge \phi'$ depends only on $\tilde{\gamma}$ and $\tilde{\phi}$. Let $\langle \tilde{\gamma}, \tilde{\phi} \rangle = \int \gamma \wedge \phi$.

e) Which harmonic forms $\varphi$ satisfy $\langle \varphi, *\varphi \rangle = 0$?

5.36. Exercise. $(x, y, z)$ is a coordinate system on $\mathbb{R}^3$ endowed with the Euclidean metric. $S_2$ is the unit sphere centered at 0 in $\mathbb{R}^3$ and $\Phi$ is the inclusion of $S_2$ in $\mathbb{R}^3$.

Let $P$ be the point of $S_2$ for which $z = 1$. When $M = (x, y, z) \in S_2$, $\theta$ will denote the angle $(\widehat{Ox, OM} - z\frac{\theta}{B_z})$, and $r$ the length of the arc $\widehat{PM}$ of meridian.
5. Riemannian Manifolds

a) What is the greatest open set $\Omega \subset S^2$ where $(\theta, r)$ is a polar coordinate system? Express the coordinates $(x, y, z)$ of $\Phi(M)$ in terms of the coordinates $(\theta, r)$ of $M \in \Omega$. From that, deduce the components of the metric $g = \Phi^*E$ on $\Omega$.

b) $(S^2, g)$ being endowed with the Riemannian connection, compute the components of the curvature tensor and of the Ricci tensor, and the value of the scalar curvature. Is the Ricci tensor proportional to the metric tensor?

c) For which values of $\theta$ are the curves $\theta = \text{Constant}$ geodesics? For which values of $r$ are the curves $r = \text{Constant}$ geodesics? What are the geodesics of the sphere?

d) Prove that the function $f = \cos r$ is an eigenfunction of the Laplacian. What is the corresponding eigenvalue? Is it the first (the smallest) nonzero eigenvalue of the Laplacian?

5.37. Exercise. a) Let $(\Omega, \Psi)$ be a local chart of a $C^\infty$ Riemannian manifold $(M, g)$. On $\Omega$, compute $\Gamma^i_{jk}$ in terms of the determinant $|g|$ of the matrix $(g_{ij})$.

b) Consider a $C^\infty$ function $\varphi$ on $\Omega$ such that $\varphi(Q) = f(r)$ for all points $Q \in \Omega$ with $r = d(P, Q)$, $P$ being a point of $\Omega$. Find a simple expression of $\Delta \varphi$.

Hint. Consider a polar geodesic coordinate system at $P$, and compute $V_i (g^{ij} \nabla_j \varphi)$.

5.38. Problem. Let $S_n$ be the unit sphere in $\mathbb{R}^{n+1}$ endowed with the Euclidean metric $E$, $0$ its center. We consider on $S_n$ the imbedded metric $g = \Phi^*E$, $\Phi$ being the inclusion $S_n \subset \mathbb{R}^{n+1}$. Let $P$ be the point with coordinates $x^j = 0$ $(1 \leq j \leq n)$, $x^{n+1} = 1$ (the $x^i$ are the coordinates on $\mathbb{R}^{n+1}$), and let $Q$ be the point opposite to $P$ on the sphere. To a point $x \in \Omega = S_n \setminus \{P\}$, we associate the point $y$, the intersection of the straight line $Px$ with the plane $\pi$ with equation $x^{n+1} = 0$: $y = \Psi(x)$. We will denote by $\{x^j\}$ $(1 \leq j \leq n+1)$ the coordinates of $x \in \mathbb{R}^{n+1}$, and by $\{y^k\}$ $(1 \leq k \leq n)$ the coordinates of $y$ in $\pi$.

a) Is $\{y^k\}$ a coordinate system on $\Omega$ corresponding to a local chart $C$?

b) Express the coordinates $\{x^j\}$ of $x \in \Omega$ in terms of the coordinates $\{y^k\}$ of $y \in \pi$.

Hint. Consider the arc $\alpha = \widehat{Qx}$ and compute in polar coordinates on $\pi$.

c) What are the components of the metric $g$ in the local chart $C$?
d) If $\mathcal{E}_\pi$ is the Euclidean metric on $\pi$, verify that $g = f(\rho)\mathcal{E}_\pi$, $f$ being a positive $C^\infty$ function of $\rho = \left[\sum_{k=1}^{n}(y^k)^2\right]^\frac{1}{2}$.

e) The straight lines of $\pi$ through 0 are geodesics for $\mathcal{E}_\pi$. Are they geodesics for the metric $g$? (Take care of the parametrization.)

f) What are the geodesics of the sphere $S_n$?

g) What is the distance $r$ from $Q$ to $x$ on $(S_n, g)$?

h) What are the components on $\Omega$ of the metric $g$ in a polar geodesic coordinate system $(r, \theta)$ with $\theta \in S_{n-1}$?

i) Let $\varphi$ be the trace on $S_n$ of the coordinate function $x^{n+1}: \varphi(x) = x^{n+1}$. Express $\varphi$ as a function of $r$.

j) Show that $\Delta \varphi = \lambda \varphi$ for a real number $\lambda$ to be determined. Thus $\varphi$ is an eigenfunction for the Laplacian $\Delta$ on $(S_n, g)$.

k) Does there exist an eigenfunction for $\Delta$ on $(S_n, g)$ equal to $\cos^2 r + k$ for some constant $k$?

For j) and k), the result of Exercise 5.37 is useful.

5.39. Problem. Let $M$ and $W$ be two connected, compact and oriented differentiable manifolds of dimension $n$, and $f$ a differentiable map of $M$ into $W$.

a) Prove that there is a real number $k$ such that, for all differential $n$-forms $\omega \in \Lambda^n(W)$,

$$\int_M f^* \omega = k \int_W \omega.$$  

Hint. Use the Hodge decomposition theorem.

b) If $f$ is not onto, show that there exists an open set $\theta \subset W$ such that $f^{-1}(\theta) = \emptyset$. Deduce that $k = 0$ under this hypothesis.

c) For the rest of this problem we suppose $f$ is onto. Let $Q \in W$ be a regular value of $f$ (the rank of $(f_*)_P$ is $n$ if $Q = f(P)$).

Prove that $f^{-1}(Q)$ is a finite set and that there exists an open neighbourhood $\theta$ of $Q$ in $W$ such that $f^{-1}(\theta)$ is the disjoint union of open sets $\Omega_i$ of $M$.

d) Prove that $k$ is an integer.

Hint. Consider a differential $n$-form $\omega$ with $\text{supp} \omega \subset \theta$, and compare $\int_W \omega$ and $\int_{\Omega_i} f^* \omega$ for each $i$.

e) Let $S_n$ be the set of unit vectors in $\mathbb{R}^{n+1}$. We endow $S_n \subset \mathbb{R}^{n+1}$ with the imbedding metric. Suppose there is on $S_n$ a differentiable field $X$ of unit tangent vectors ($g(X, X) = \mathcal{E}(X, X) = 1$).

Let $X^i(x)$ be the components of $X$ in $\mathbb{R}^{n+1}$. We consider the map $F_t$ of $S_n$ into $\mathbb{R}^{n+1}$: $x \rightarrow y$ of coordinates $y^j = x^j \cos \pi t + X^j(x) \sin \pi t$,
where the $x^j$ are the coordinates of $x \in \mathbb{R}^{n+1}$. Verify that $F_t$ is a differentiable map of $S_n$ into $S_n$. We denote by $k(t)$ the number defined in a) for the map $F_t$. Show that $t \rightarrow k(t)$ is continuous.

f) What are the values of $k(0)$ and $k(1)$? When $n$ is even, prove that there does not exist a vector field on $S_n$ which is nowhere zero.

5.40. Exercise. Let $(M_n, g)$ be a $C^\infty$ Riemannian manifold and $P \in M$. For two vectors $X, Y$ of $\mathbb{R}^n$, show that for any $\epsilon > 0$, there exists $\eta > 0$ such that if $0 \leq t \leq \eta$, then

$$d(\exp_P tX, \exp_P tY) \leq (1 + \epsilon)\|X - Y\|t.$$ 

5.41. Problem. Let $(M_n, g)$ be a complete $C^\infty$ Riemannian manifold of dimension $n \geq 2$. We admit that for every pair of points $P$ and $\tilde{P}$ of $M$, there exists at least one geodesic $C$ through $P$ and $\tilde{P}$ whose length is $d(P, \tilde{P})$. So let $C$ be a geodesic from $P$ to $\tilde{P}$ whose length

$$L(C) = d(P, \tilde{P}) = a : \mathbb{R} \ni [0, a] \ni t \rightarrow C(t) \in M.$$ 

$\tilde{P} = C(a)$, and $t$ is the arc length parameter. At $P = C(0)$, we consider an orthonormal basis $\{e_i\}$ $(i = 1, 2, \cdots, n)$ of $T_P(M)$ such that $e_1 = (\frac{dC}{dt})_{t=0}$. We denote by $e_i(t)$ the parallel translated vector of $e_i$ from $P$ to $C(t)$ along $C$.

a) Verify that the vectors $\{e_1(t), e_2(t), \cdots, e_n(t)\}$ form an orthonormal basis of $T_{C(t)}(M)$.

b) In the sequel we consider, on a neighbourhood $\Omega$ of $C$, a coordinate system $\{x^i\}$ such that $C(t)$ has $(t, 0, 0, \cdots, 0)$ for coordinates, and such that $(\frac{\partial}{\partial x^j})_{C(t)} = e_j(t)$ for all $t$ and $j$.

What can we deduce from this property for the components $g_{ij}(C(t))$ of $g$ and for the Christoffel symbols $\Gamma^k_{ij}(C(t))$?

c) Let $B$ be the unit ball in $\mathbb{R}^{n-1}$, and consider the map $f$ of $[0, a] \times B$ into $M$ defined by

$$(t, \xi) \rightarrow f(t, \xi) = \exp_{C(t)} \left[ \sum_{i=2}^n \xi^i \hat{e}_i(t) \right],$$

where $\{\xi^i\} (2 \leq i \leq n)$ are the components of the vector $\xi \in \mathbb{R}^{n-1}$, and where $\hat{e}_i(t) = (\partial/\partial x^i)_0$, $\{x^i\}$ being the coordinates of $\mathbb{R}^n$. What is the differential of $f$ at $(t, 0)$? Is it invertible?

d) For a given point $Q$ of $M$, verify that there exists at least one point $\hat{Q} \in C$ such that $d(Q, \hat{Q}) = d(Q, C)$, where by definition

$$d(Q, C) = \inf\{d(Q, R) \mid R \in C\}.$$
e) Prove that if \( \tilde{Q}, \) one of the points found in d), is neither \( P \) nor \( \tilde{P}, \) then a geodesic \( \gamma \) from \( Q \) to \( \tilde{Q}, \) of length \( L(\gamma) = d(Q, \tilde{Q}), \) is orthogonal to \( C \) at \( \tilde{Q}. \)

**Hint.** For two nonorthogonal vectors \( X, Y \in T_Q(M) \) \((g_Q(X, Z) \neq 0), \) observe that we can choose a vector \( Y \) proportional to \( Z \) so that \( g_Q(X - Y, X - Y) < g(X, X). \) Use the result of Exercise 5.40.

f) Prove that there is \( \varepsilon > 0 \) such that, if \( d(Q, C) < \varepsilon, \) then there exists a unique point \( \tilde{Q} \) of \( C \) such that \( d(Q, \tilde{Q}) = d(Q, C). \)

**g)** Deduce from f) that \( (t, \xi^1, \xi^2, \ldots, \xi^n) \) is a coordinate system on \( \theta = f([0, a[ \times B_1), \) \( B_1 \) being the open ball of radius \( \varepsilon \) in \( \mathbb{R}^{n-1}. \)

**h)** In this coordinate system, compute the Christoffel symbols at \( C(t). \)

**5.42. Problem.** Let \((x, y, z)\) be a coordinate system for \( \mathbb{R}^3 \) and \((u, v)\) a coordinate system for \( \mathbb{R}^2. \) For two real numbers \( a \) and \( b \) with \( a > b > 0 \) we define the map \( \psi \) of \( \mathbb{R}^2 \) into \( \mathbb{R}^3 \) as follows:

\[
\begin{align*}
x &= (a + b \cos v) \cos u, \\
y &= (a + b \cos v) \sin u, \\
z &= b \sin v.
\end{align*}
\]

**a)** What is the rank of \( \psi? \)

**b)** We identify \( C \times C \) (\( C \) the circle of radius 1) to the quotient of \( \mathbb{R}^2 \) by the equivalence relation

\[
(u, v) \sim (\tilde{u}, \tilde{v}) \leftrightarrow \begin{cases} 
\tilde{u} - u = 2k\pi, \quad k \in \mathbb{Z}, \\
\tilde{v} - v = 2h\pi, \quad h \in \mathbb{Z}.
\end{cases}
\]

Show that we can define a map \( \tilde{\psi} \) of \( C \times C \) into \( \mathbb{R}^3 \) from the map \( \psi. \) Is \( \tilde{\psi} \) an imbedding?

**c)** Find a polynomial \( P \) of the fourth degree in \( x, y, z \) which vanishes on \( M = \psi(\mathbb{R}^2) \) and only on \( M. \)

**d)** Prove that \( M \) is a submanifold of \( \mathbb{R}^3. \) Is \( M \) diffeomorphic to \( C \times C? \)

**e)** Determine the Riemannian metric \( g \) on \( M \) induced by the Euclidean metric on \( \mathbb{R}^3. \) Verify that the nonzero components of \( g \) (in the coordinate system \( u, v \)) are equal to \( b^2 \) and \((a + b \cos v)^2.\)

Compute the Christoffel symbols of the Riemannian connection.

**f)** Compute the scalar curvature \( R \) of \((M, g).\) Note that it has the same sign as \( \cos v. \) What is the value of \( \int_M R dV? \)

**g)** Write the differential equations that arcs of geodesics \( t \rightarrow (u(t), v(t)) \) satisfy on \((M, g).\)

Are the intersections of \( M \) with the planes through the axis \( Oz \) arcs of geodesics?
Find all geodesics which are included in a plane perpendicular to \( O_2 \).

h) Show that, if at a point of the arc of geodesic \( \gamma \), \( \frac{d\gamma}{dt} \) does not vanish, then \( \frac{d\gamma}{dt} \) is not zero at any point of \( \gamma \).

*Hint.* Use Cauchy’s theorem.

i) Deduce from the result above that we can substitute the parameter \( t \) for \( u \), except for some geodesics \( \gamma \) that one will identify.

What is the differential equation (E), where the unknown is \( u(t) \), satisfied by the differentiable curves which are arcs of some geodesic of \( M \) different from the geodesic \( \gamma \) found above.

Verify that \( \frac{d^2 u}{dt^2} \) always has the same sign as \( -\sin v \).

j) We study the geodesics from \( P = \psi(0,0) \). What is the geodesic for which \( \left( \frac{d\gamma}{dt} \right)_P = 0 \)? Which symmetry permutes the geodesic for which \( \left( \frac{d\gamma}{dt} \right)_P = \alpha > 0 \) with the one for which \( \left( \frac{d\gamma}{dt} \right)_P = -\alpha \)?

We denote by \( u_\alpha \) the solution of equation (E) with \( u_\alpha(0) = 0 \) and \( u'_\alpha(0) = \alpha \).

k) Prove that if \( \alpha \) is small enough (\( 0 < \alpha << 1 \)), then \( du_\alpha/du \) vanishes for the first time at a value \( u_\alpha < \pi/2 \) when \( u \) increases from zero. What is the sign of \( u_\alpha(u_\alpha) \)?

l) Show that \( u_\alpha(2u_\alpha) = 0 \). Deduce that there are at least three geodesics from \( P \) to \( Q = \psi(2u_\alpha,0) \). Compare their lengths to \( \pi \sqrt{b(a + b)} \) and \( (a + b)\pi \).

5.43. Problem. Let \( M_n \) be a \( C^\infty \) differentiable submanifold of \( R^{n+k} \) \( (k \geq 1) \). We denote by \( F \) the inclusion \( M_n \to R^{n+k} \).

By \( F_* \) we identify \( X \in T(M) \) to \( F_\ast X \in T(R^{n+k}) \); thus \( T(M) \subset T(R^{n+k}) \).

Let \( E \) be the Euclidean metric on \( R^{n+k} \), and set \( g = F^\ast E \).

\( X, Y \) and \( \nu \) will be three vector fields defined on \( M \), with \( X \) and \( Y \) included in \( T(M) \).

a) Consider three vector fields \( \tilde{X}, \tilde{Y} \) and \( \tilde{\nu} \) in \( R^{n+k} \) which are equal to \( X, Y \) and \( \nu \) on \( M \). For the Riemannian connection \( \tilde{D} \) on \( (R^{n+k}, E) \), verify that \( \tilde{D}_{\tilde{X}} \tilde{\nu} /M \) depends only on \( X \) and \( \nu \).

b) For \( Q \in M \), set \( (D_X Y)_Q = \pi_Q(D_{\tilde{X}} \tilde{Y})_Q \), where \( \pi_Q \) is the orthogonal projection of \( T_Q(R^{n+k}) \) on \( T_Q(M) \).

Compare \( [\tilde{X}, \tilde{Y}] /M \) with \( [X, Y] \).

Verify that \( (X, Y) \to D_X Y \) is a connection on \( M \), which is the Riemannian connection of \( (M, g) \).

c) In a neighbourhood \( \theta \) of \( P \in M \), we consider \( k \) orthonormal vector fields \( \nu_i \) \( (i = 1,2, \cdots , k) \) orthogonal to \( M \) \( (E(\nu_i, \nu_j) = \delta_i^j \), and if \( Q \in \theta \), then \( \pi_Q \nu_i = 0 \). Set \( \ell_i(X, Y) = E(\tilde{D}_X \nu_i, Y) \).
Establish the following formula:
\[ \mathcal{D}_X Y = D_X Y - \sum_{i=1}^{k} \ell_i(X, Y) \nu_i. \]

Is \( \ell_i(X, Y) \) a symmetric bilinear form on \( T_Q(M) \)?

d) Compute \( \mathcal{D}_X \mathcal{D}_Y Z \), and then show that
\[ \varepsilon(\mathcal{D}_X \mathcal{D}_Y Z, T) = g(D_X D_Y Z, T) - \sum_{i=1}^{k} \ell_i(X, T) \ell_i(Y, Z) \]
for two vector fields \( Z < T \in M \) included in \( T(M) \).

e) Deduce from this formula that the curvature tensor of \((M, g)\) is given by
\[ R(X, Y, Z, T) = \sum_{i=1}^{k} [\ell_i(X, Z) \ell_i(Y, T) - \ell_i(X, T) \ell_i(Y, Z)]. \]

**Hint.** Use the fact that the curvature tensor of the Euclidean metric vanishes.

f) Suppose \( k = 1 \), and let \( \lambda_j \) \((j = 1, 2, \ldots, n)\) be the eigenvalues of \( \ell_1 \) at \( Q \).
If \( n = 2 \), compute the scalar curvature of \((M, g)\) at \( Q \) in terms of the \( \lambda_i \). If \( n > 2 \), prove that the sectional curvature of \((M, g)\) at \( Q \) cannot be negative with respect to all 2-planes of \( T_Q(M) \).

**5.44. Problem.** Let \((M, g)\) be a \( C^\infty \) compact Riemannian manifold of dimension \( n \) and let \( X, Y \) be \( C^\infty \) vector fields on \( M \). We consider the one parameter local group \( G \) of local diffeomorphisms \( \varphi_t(x) \) corresponding to \( X \).

a) For a covariant tensor field \( h \), verify that \( d(\varphi_t^* h)/dt = \varphi_t^* L_X h \). Deduce from this that a necessary and sufficient condition for \( h \) to be invariant under \( G \) is \( L_X h = 0 \).

By definition, \( h \) is invariant under \( G \) if \( \varphi_t^* h = h \) when \( \varphi_t^* h \) exists.

b) A \( C^\infty \) differentiable map \( \psi \) of \( M \) into \( M \) is called a conformal transformation of \((M, g)\) if \( \psi^* g = e^f g \), \( f \) being a \( C^\infty \) function on \( M \). Set \( h = |g|^{-\frac{1}{n}} h \), where \( |g| \) is the determinant of \((g_{ij})\). Compute \( |\psi^* g| \) and \( \psi^* h \).

Show that \( L_X h = 0 \) is a necessary and sufficient condition for the diffeomorphisms of \( G \) to be conformal transformations of \((M, g)\).

c) Let \( P \) be a point where \( Y(P) \neq 0 \). Consider a local chart \((\Omega, \gamma)\) at \( P \), with \( \gamma(\Omega) \) a ball centered at \( \gamma(P) = 0 \in \mathbb{R}^n \). We choose \((\Omega, \gamma)\) such that \( Y \neq 0 \) on \( \Omega \). \( \{x^i\} \) being the associated coordinate system, we suppose \( Y^i \neq 0 \) on \( \Omega \) \((Y = Y^i \partial/\partial x^i)\). Let \( \psi_t \) be the one parameter group of local diffeomorphisms corresponding to \( Y \) and let \( x_0 \) be a
point in \( \Omega \) such that \( x_0 = \{ x^1, x^2, \ldots, x^{n-1}, 0 \} \). If it exists in \( \Omega \), we set \( x = \psi_t(x_0) \). Prove that \( x^1, \ldots, x^{n-1}, t \) form a coordinate system in a neighbourhood of \( P \).

(d) Prove that \( \mathcal{L}_Y h = |g|^{-\frac{1}{2}}[\mathcal{L}_Y g - \frac{2}{n} \nabla_i Y^i g] \).

*Hint.* Use the coordinate system constructed above.

**5.45. Problem.** Let \((M, g)\) be a connected complete \( C^\infty \) Riemannian manifold of dimension \( n \).

If \( W \) is a submanifold of \( M \), we consider the tangent vectors to \( W \) as tangent vectors of \( M \) (each \( X \in T_W(W) \), \( Q \in W \), is identified to \((i_*)_QX\), where \( i : W \rightarrow M \) is the inclusion). Moreover, \( W \) is endowed with the metric \( \tilde{g} = i^* g \).

(a) Since the manifold is complete, prove that \( \text{exp}_P X \) exists for any \( X \in \mathbb{R}^n \) and any \( P \in M \).

*Hint.* Consider a sequence \{\( t_i \)\} \( \subset \mathbb{R} \), which converges to \( \tau \), such that \( Q_i = \text{exp} t_i X \) exists in \( M \) for any \( i \). Prove that the sequence \{\( Q_i \)\} in \( M \) converges to a point \( Q \in M \). Then apply at \( Q \) the theorem of the exponential mapping (5.11).

(b) The compact submanifold \( W \) is said to be **totally geodesic** if any geodesic of \( W \) is a geodesic of \( M \). If \( W \) has this property, prove that a geodesic \( C \) of \( M \) is included in \( W \) if at one of its points \( C(t) \), \((\frac{dC}{dt})_{C(t)} \in T(W) \).

(c) We admit that there exists a coordinate system \{\( x^i \)\} in a neighbourhood of a geodesic arc \( C : [0, \tau] \ni t \rightarrow C(t) \in M \), such that the coordinates of the points \( C(t) \) are \( x^1 = t, x^i = 0 \) for \( i > 1 \), and such that this coordinate system is normal at each point \( C(t) \) of \( C \). Prove that the vector fields \((\partial/\partial x^i)\) are parallel along \( C \).

(d) Let \( W_p \) and \( V_q \) be two compact totally geodesic submanifolds of dimension respectively \( p \) and \( q \) such that \( W_p \cap V_q = \emptyset \). Prove that there exist \( P_0 \in W \) and \( Q_0 \in V \) such that \( d(P_0, Q_0) = d(W, V) \). By definition \( \tau = d(W, V) = \inf d(x, y) \) for all \( x \in W \) and \( y \in V \).

(e) We admit that there exist a geodesic \( C \) from \( P_0 \) to \( Q_0 \) (\( P_0, Q_0 \) as in d)) of length \( L(C) = \tau \), and a coordinate system \{\( x^i \)\} as in c). Set \( P_0 = C(0) \) and \( Q_0 = C(\tau) \). Let \( H \) be the set of the vectors of \( T_{Q_0}(M) \) obtained from the vectors of \( T_{P_0}(W) \) by parallel transport along \( C \).

For the rest of this problem we suppose \( p + q \geq n \). Prove that \( H \cap T_{Q_0}(V) \) is not reduced to the zero vector. Let \( Y \in H \cap T_{Q_0}(V) \) be of norm 1 \((g_{Q_0}(Y, Y) = 1)\).
f) Consider the vector field $X(t)$ parallel along $C$ such that $X(\tau) = Y$. For a chosen $\lambda \in \mathbb{R}$, we define the arc $\gamma_\lambda$ by

$$[0, \tau] \ni t \rightarrow \gamma_\lambda(t) = \exp_{C(t)} \lambda X(t).$$

Verify that $\gamma_\lambda$ is a $C^\infty$ differentiable arc whose endpoints are a point of $W$ and a point of $M$. What can we say about $f(\lambda) = L(\gamma_\lambda)$?

g) Compute first $f'(0)$, then $f''(0)$, in terms of the sectional curvature $\sigma(X(t), \frac{dC'}{dt})$. We recall that the curvature tensor is given by

$$R_{klij}(C(t)) = \frac{1}{2}(\partial_{ik}g_{jl} - \partial_{ik}g_{jl} - \partial_{ij}g_{kl} + \partial_{jk}g_{il})C'(t)$$

in a coordinate system which is normal at $C(t)$.

h) Deduce from g) that the intersection of $W_p$ and $V_q$ is not empty if the sectional curvature is everywhere strictly positive.

5.46. Problem. Let $(M, g)$ be a compact Riemannian manifold, and consider a Riemannian cover $(W, \tilde{g})$ of $(M, g)$ such that $W$ is simply connected. If $\pi : W \rightarrow M$ is the projection of $W$ on $M$, then $\tilde{g} = \pi^*g$. We suppose that $W$ has more than one sheet, that is to say, $(M, g)$ is not simply connected.

a) Let $\tilde{C}$ be a differentiable arc of $W$. $C = \pi(\tilde{C})$ its projection. Show that $\tilde{L}(\tilde{C}) \geq L(C)$ and that the projection of a geodesic arc is a geodesic arc ($L$ denotes the length).

b) Verify that $(W, \tilde{g})$ is complete.

c) Let $C : [a, b] \rightarrow M$ be a differentiable closed curve of $M$ which is not homotopic to zero. Set $P = C(a) = C(b)$ and choose a point $\tilde{P} \in \pi^{-1}(P)$. Show that we can construct by continuity a differentiable curve $\tilde{C} \subset W$, locally diffeomorphic to $C$ by $\pi/\tilde{C}$, such that $\tilde{C}(a) = \tilde{P}$. Why is $\tilde{C}(b) = \tilde{P}$ not the point $P$?

d) If $C_0$ is a curve from $P$ to $P$, homotopic to $C$, prove that we find the same endpoint $\tilde{P}' = \tilde{C}_0(b)$. $\tilde{C}_0$ being constructed from $C_0$ as $\tilde{C}$ from $C$ with $\tilde{C}_0(a) = \tilde{P}$. We set $f(P) = d(\tilde{P}, \tilde{P}')$. So $f(P)$ depends on $P$ and on the chosen homotopy class.

e) For a point $R$ of $M$, we consider a differentiable curve $\gamma : [\alpha, \beta] \rightarrow M$ homotopic to $C$ and such that $\gamma(\alpha) = \gamma(\beta) = R$. For instance, if $\Gamma$ is a differentiable curve from $R$ to $P$, $\gamma$ may be a regularization of the following curve: $\Gamma$, then $C$, then $\Gamma$ from $P$ to $R$.

From a point $\tilde{R} \in \pi^{-1}(R)$, we construct (as in c) and d)) a curve $\tilde{\gamma} \subset W$ locally diffeomorphic to $\gamma$ by $\pi/\tilde{\gamma}$ with $\gamma(\alpha) = \tilde{R}$. Set $\tilde{R}' = \tilde{\gamma}(\beta)$ and $f(R) = d(\tilde{R}, \tilde{R}')$.

Prove that $M \ni R \rightarrow f(R) \in \mathbb{R}$ is a strictly positive continuous map which achieves its minimum at least at one point $Q \in M$. 

f) As above, we construct the point $Q'$ from a point $Q \in \pi^{-1}(Q)$. Then we consider a minimizing geodesic $\tilde{\gamma}$ from $Q$ to $Q'$. Show that $\pi(\tilde{\gamma})$ is a geodesic from $Q$ to $Q$, then a closed geodesic (the curve is also a geodesic at $Q$). What is its length?

We assume that any pair of points $(P, Q)$ of a complete Riemannian manifold $(M, g)$ may be joined by a minimizing geodesic $\gamma$ (that is to say, $L(\gamma) = d(P, Q)$).

5.47. Exercise. Let $(M, g)$ be a compact Riemannian manifold. Consider the metric $g'$ on $M$ defined by $g' = e^{f}g$, where $f$ is a $C^\infty$ function on $M$.

a) Show that $(M, g')$ is a Riemannian manifold.

b) In a normal coordinate system at $P \in M$, compute the Christoffel symbols $\Gamma^j_{ik}(P)$ of $(M, g')$ in terms of $f$.

c) In a neighbourhood of $P$, deduce the expression of $C^j_{ik} = \Gamma^j_{ik} - \Gamma^j_{ik}$, where $\Gamma^j_{ik}$ are the Christoffel symbols of $(M, g)$.

d) In terms of $f$, compute $R^{}_{lijk} - R^{}_{lijk}$, where $R^{}_{lijk}$ and $R^{}_{lijk}$ are respectively the components of the curvature tensors of $(M, g')$ and of $(M, g)$.

e) Deduce, from the result above, the expression of $R^{}_{ijkl} - R^{}_{ijkl}$, the difference of the components of the Ricci tensors of $(M, g')$ and $(M, g)$.

f) Express $R'$ as a function of $R$ and $f$, $R'$ and $R$ being the scalar curvatures of $(M, g')$ and $(M, g)$.

5.48. Problem. Let $(M, g)$ be a Riemannian manifold with nonpositive sectional curvature $(\sigma(X, Y) \leq 0)$. Given a geodesic $C : \mathbb{R} \ni [a, b] \ni t \rightarrow C(t) \in M$, we consider an orthonormal frame $e_1, e_2, \ldots, e_n$ of $T_p(M)$ ($P = C(a)$) such that $e_1 = (\frac{dC}{dt})_a$. Then at $Q = C(t)$ we consider the vectors $e_i(t)$ obtained from $e_i$ by parallel translation along $C$. We suppose that $t$ is the arc length.

a) Do the vectors $e_i(t)$ ($i = 1, 2, \ldots, n$) form a basis of $T_Q(M)$?

b) Let $x^1, x^2, \ldots, x^n$ be a coordinate system in a neighbourhood of $C$ such that $(\partial/\partial x^i)_Q = e_i(t)$. What are the values of the components of the metric tensor at $Q$, and what can we say about the Christoffel symbols at $Q$?

c) We suppose that there exists a coordinate system in a neighbourhood $\Omega$ of $C$ such that at any point $Q$ of $C$ we have $g_{ij}(Q) = \delta^j_i$ and $\Gamma^k_{ij}(Q) = 0$. Along $C$, we consider a vector field $X$ whose components $X_i(t)$ written for $X^i(t)$ at $C(t)$ satisfy

$$\frac{d^2 X^i(t)}{dt^2} = -R^i_{lijk}(C(t))X^j(t).$$
Show that such vector fields $J$ exist, and that they form a vector space $E$. What is the dimension of $E$?

d) Verify that there exist vector fields $J$ which are orthogonal at each point to the geodesic, and that they form a vector subspace of $E$.

e) Prove that a nonzero vector field $J$ may be zero only once.

f) If $X$ and $Y$ are two vector fields $J$ along $C$ vanishing at $P$, show that $g(Y, X') = g(Y', X)$, where $X' = \frac{dX}{dt}$.

g) Prove that there exists one and only one vector field $J$ such that $X(a) = 0$ and $X(b) = X_0$, a given vector of $T_{C(b)}(M)$.

h) Consider $n-1$ vector fields $J$ which satisfy $Y_i(a) = 0$ and $Y_i(b) = c_i(b)$ ($i = 2, 3, \ldots, n$). Verify that $n-1$ differentiable functions $f_i(t)$ on $[a, b]$ are associated to a differentiable vector field $Z$ along $C$ vanishing at $P$ and orthogonal to $C$ such that $Z(t) = \sum_{i=2}^{n} f_i(t)Y_i(t)$.

i) We define

$$ I(Z) = \int_{a}^{b} \left\{ g(Z', Z') - g\left[ R\left( \frac{dC}{dt}, Z, \frac{dC}{dt} \right) \right] \right\} dt. $$

Set $X(t) = \sum_{i=2}^{n} f_i(b)Y_i(t)$, and show that $I(X) = g(X'(b), X(b))$.

j) Prove that $I(Z) \geq I(X)$.

**Hint.** Consider $W(t) = \sum_{i=2}^{n} f_i(t)Y_i(t)$, and establish the equality

$$ I(Z) = \int_{a}^{b} g(W(t), W(t))dt + g(X'(b), X(b)). $$

5.49. **Problem.** Let $(M_n, g)$ be a connected complete $C^\infty$ Riemannian manifold of dimension $n$, and $f$ a map of $M$ into $M$ which preserves the distance: $d(P, Q) = d(f(P), f(Q))$ for all pairs $(P, Q) \in M \times M$.

We assume that any pair of points $(P, Q)$ can be joined by a geodesic:

a) Show that the image by $f$ of a geodesic arc is a geodesic arc. Recall that if $d(P, Q) < \lambda(P)$, the injectivity radius at $P$, then there exists a unique geodesic arc from $P$ to $Q$ whose length is $d(P, Q)$.

b) If $X$ and $Y$ belong to $\mathbb{R}^n$, prove that

$$ \lim_{t \to 0} [d(\exp_P tX, \exp_P tY)t^{-1}] = \|X - Y\|. $$

c) Prove that any geodesic arc through $P$ can be extended infinitely (that is to say, $\exp_P X$ exists for any $X \in \mathbb{R}^n$).

d) The image by $f$ of a geodesic through $P$ ($\tilde{X}$ is its tangent vector at $P$) is a geodesic through $P' = f(P)$. We denote by $X' = \varphi(\tilde{X})$ its tangent vector at $P'$.

Show that $\varphi$ is a linear map of $T_P(M)$ onto $T_{P'}(M)$.
5. Riemannian Manifolds

e) Establish that \( f \) is differentiable and that \( f^*(g \circ f) = g \). \( f \) is called an isometry.

f) Prove that two isometries \( f \) and \( h \) are identical if there exists a point \( P \in M \) such that \( f(P) = h(P) \) and \( (f_*)_P = (h_*)_P \).

g) Define a map of \( I(M) \), the set of the isometries of \( M \), into \( M \times O_n \). What can we say when \( M \) is the sphere \( S_n \)?

5.50. Problem. Let \( (M_n, g) \) be a compact Riemannian manifold of dimension \( n \), and \( X, Y \) two \( C^\infty \) vector fields on \( M_n \). Consider the differential 1-forms \( \xi \) and \( \alpha \) associated to \( X \) and \( Y \); \( \xi_i = g_{ij}X^j \) and \( \alpha_i = g_{ij}Y^j \) are their components in a local chart.

We set \( h = |g|^{-\frac{1}{2}}g \), where \( |g| \) is the determinant of \( (g_{ij}) \).

a) Show that, in a neighbourhood of a point where \( Y \) is nonzero, there exists a coordinate system \( \{x^i\} \) such that \( [Y, \partial/\partial x^i] = 0 \) for all \( i \).

b) Deduce from a) that \( \mathcal{L}_Y h = |g|^{-\frac{1}{2}}[\mathcal{L}_Y g - \frac{2}{n} \nabla_i Y^i g] \).

c) Set \( t(\alpha) = |g|^{\frac{1}{2}} \mathcal{L}_Y h \). Show that the components of \( t(\alpha) \) are \( t(\alpha)_{ij} = \nabla_i \alpha_j + \nabla_j \alpha_i + \frac{2}{n} \delta \alpha_{ij} \).

d) Verify that \( (\Delta \alpha)_j = -\nabla^i \nabla_i \alpha_j + R_{ij} \alpha^i \), where \( R_{ij} \) are the components of the Ricci tensor.

e) Prove that \( \nabla^i t(\alpha)_{ij} = 2R_{ij} \alpha^i - (\Delta \alpha)_j - (1 - \frac{2}{n})(d\alpha)_{ij} \).

f) Consider the differential 1-form \( u(\alpha) \) defined by \( u(\alpha)_i = \alpha^j t(\alpha)_{ij} \). Compute \( \delta u(\alpha) \) in terms of \( \alpha^j \nabla^i t(\alpha)_{ij} \) and \( (t(\alpha), t(\alpha)) \).

g) Show that \( \mathcal{L}_X h = 0 \) if and only if

\[
(n - 1)\Delta \xi + \left( 1 - \frac{2}{n} \right) \delta d\xi = S(\xi),
\]

where \( S(\xi)_i = nR_{ij} \xi^j \).

h) Prove that \( \mathcal{L}_X g = 0 \) if and only if

\[
\nabla^i \nabla_i \xi_j + R_{ij} X^i = 0 \quad \text{and} \quad d\delta \xi = 0.
\]

i) Compute \( \delta S(\xi) \). Deduce from the result that if \( \mathcal{L}_X h = 0 \) and if the scalar curvature \( R \) is constant, then

\[
(n - 1)\Delta \delta \xi = R \delta \xi.
\]

5.51. Exercise. Let us consider the unit sphere \( S_n \subset \mathbb{R}^{n+1} \) endowed with the canonical metric \( g = i^*\mathcal{E} \) \( (i \) the inclusion), and let \( C \) be the geodesic of \( S_n \) through \( x \in S_n \) with initial condition \( (x, v) \), \( v \) belonging to \( T_x(S_n) \). Consider the orthogonal symmetry \( \sigma \) with respect to the 2-plane \( P \) defined by \( x \) and \( v \) in \( \mathbb{R}^{n+1} \).

a) Prove that \( C \) is included in \( P \).
b) Deduce that the geodesics of the sphere are the great circles.

c) What are the geodesics of the quotient $P_n(\mathbb{R})$ of $S_n$ by the antipodal map? Show that they are periodic with period $\pi$ if they are parametrized by arc length.

5.52. Exercise. Let us consider the unit sphere $S_{2n+1} \subset \mathbb{C}^{n+1}$ endowed with the canonical metric $g$. The complex projective space $P_n(\mathbb{C})$ is the quotient of $S_{2n+1}$ by the equivalence relation $\mathcal{R}$ (see 1.41); let $q: S_{2n+1} \to P_n(\mathbb{C})$ be the corresponding projection. Let $z \in S_{2n+1}$ and $v \in T_z(S_{2n+1})$. We consider the curve $C : [0, \pi] \ni t \to z \cos t + v \sin t$.

a) Verify that $C \subset S_{2n+1}$ and that $q(C)$ is a smooth closed curve in $P_n(\mathbb{C})$.

b) Prove that there is a unique Riemannian metric $\bar{g}$ on $P_n(\mathbb{C})$ such that $q^* \bar{g} = g$.

c) What is the length of $q(C)$ in $(P_n(\mathbb{C}), \bar{g})$?

5.53. Problem. Let $M$ be a $C^\infty$ Riemannian manifold and let $C$ be a geodesic from $P$ to $Q$ ($[0, r] \ni t \to C(t) \in M$). According to Problem 5.41, there exists a coordinate system $\{x^i\}$ on a neighbourhood $\theta$ of $C$ which is normal at each point of $C$ (the coordinates of $C(t)$ are $(t, 0, 0, \ldots, 0)$). For $|\lambda| < \epsilon$, let us consider a family $\{C_{\lambda}\}$ of differentiable curves in $\theta$ defined by the $C^2$ maps $[0, r] \times (-\epsilon, \epsilon) \ni (t, \lambda) \to x^i(t, \lambda) \in \theta$ with $x^i(0, \lambda) = 0$, $x^1(t, 0) = t$ and $x^i(t, 0) = 0$ for $i > 1$.

a) Verify that the first variation of the length integral of $C_{\lambda}$ is zero at $\lambda = 0$.

b) Prove that the second variation of the length integral of $C_{\lambda}$ at $\lambda = 0$ is

\[
\left( \frac{d^2L(C_{\lambda})}{d\lambda^2} \right)_{\lambda=0} = \int_0^r \left[ \sum_{i=2}^n \left( \frac{dy^i}{dt} \right)^2 + R_{1i1j}(C(t))y^i(t)y^j(t) \right] dt = 0,
\]

where $y^i(t) = (\partial x^i(t, \lambda)/\partial \lambda)_{\lambda=0}$.

5.54. Problem. Prove that a compact orientable Riemannian manifold of even dimension with positive sectional curvature is simply connected.

Hint. Use the result of Problem 5.46 and the expression of the second variation of the length integral found in the problem above.

5.55. Exercise. Let $(M_n, g)$ be a Riemannian manifold, endowed with the Riemannian connection $D$, and $\{x^i\}$ ($i = 1, 2, \ldots, n$) a normal coordinate system on $P \in M$ corresponding to the local chart $(\Omega, \varphi)$. In $\mathbb{R}^n$, with coordinates $\{w^i\}$, we consider the ball $B_\epsilon$ of center 0 $\in \mathbb{R}^n$ and radius $\epsilon > 0$, and a $C^\infty$ map $\phi$ of $B_\epsilon$ into $\Omega$ (with $\phi(0) = P$) such that $t \to \phi(t, 0, 0, \cdots, 0)$ is a geodesic on $M$. Define $z = (w^3, w^4, \ldots, w^n)$, and
let $Z(t, u, z) = \partial \phi(t, u, z)/\partial u$ be the tangent vector at $\phi(t, u, z)$ to the curve $u \rightarrow \phi(t, u, z)$ and $T(t, u, z) = \partial \phi(t, u, z)/\partial t$ the tangent vector to the curve $t \rightarrow \phi(t, u, z)$.

a) Compute the bracket $[Z(t, u, z), T(t, u, z)]$.

b) Set $X(t) = Z(t, 0, 0)$ and $Y(t) = T(t, 0, 0)$. Show that $D_Y D_Y X = R(Y, X)Y$.

c) Prove that $g(X(t), Y(t)) = at + b$, $a$ and $b$ being two real numbers.

5.56. Exercise. Let $M_n$ be a compact $C^\infty$ orientable manifold of dimension $n > 1$. We consider a Riemannian metric $g$ on $M_n$ and $\eta \in \Lambda^n(M)$ defined in a coordinate system by

$$\eta = \sqrt{|g|} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

where $|g| = \det((g_{ij}))$.

a) Verify that $\eta$ is a differential $n$-form on $M$.

b) We endow $M_n$ with a linear connection whose Christoffel symbols are $\Gamma^k_{ij}$. Show that a necessary and sufficient condition for $\nabla \eta = 0$ is that $\Gamma^j_i = \frac{1}{2} \partial_i \log |g|$.

c) Does the Riemannian connection satisfy this condition?

d) We suppose (here and for e)) that the manifold is endowed with the Riemannian connection. Let $X$ be a vector field. Show that

$$\nabla_i X^i = \frac{1}{\sqrt{|g|}} \partial_i (X^i \sqrt{|g|}).$$

e) Let $\alpha = i(X)\eta$. Compute $d\alpha$ and the value of $\int_M \nabla_i X^i \eta$.

5.57. Exercise. Let $M_n$ be a complete Riemannian manifold of dimension $n > 1$. For $X \in S_{n-1}$ we define

$$\mu(X) = \sup\{r \in \mathbb{R} \mid d[\exp_P(rX), P] = r\},$$

where $S_{n-1} = \{X \in \mathbb{R}^{n-1} \mid |X| = 1\}$ and $P$ is a given point.

a) Show that $X \rightarrow \mu(X)$ is a continuous map of $S_{n-1}$ into $I = \mathbb{R}^+ \cup \{+\infty\}$, endowed with the usual topology.

b) Prove that for $M_n$ to be compact, it is necessary and sufficient that $\mu(X)$ be finite for all $X \in S_{n-1}$.

c) If $M_n$ is compact and has non-positive sectional curvature, show that, given a pair $(P, Q)$ of points of $M$, there exist only a finite number of minimizing geodesics from $P$ to $Q$. Then show that the set $C_P = \{\exp_P[\mu(X)X] \mid X \in S_{n-1}\}$ is the union of a finite number of manifolds of dimension $n - 1$. We will assume (or prove) that, when $Q \in C_P$, if there is only one minimizing geodesic from $P$ to
Solutions to Exercises and Problems

Solution to Exercise 5.29.

By hypothesis $R_{ij}(P) = f(P)g_{ij}(P)$. Contracting this equality gives $R(P) = nf(P)$ ($g_{ij}g^{kj} = \delta^k_i$; if we do $i = k = 1, 2, \ldots, n$, we find that $g_{ij}g^{ij} = n$). On the other hand, contracting twice the second Bianchi identity (5.8) yields (we multiply it by $g^{im}$, then by $g^{il}$)

$$\nabla^j R_{ijkl} + \nabla_k R_{il} - \nabla_l R_{ik} = 0; \text{ then } \nabla_k R = 2\nabla^i R_{ik}.$$

Indeed, for instance,

$$g^{im}\nabla_k R_{ijlm} = \nabla_k (g^{im} R_{ijlm}) = \nabla_k R_{ik}$$

and

$$g^{il}\nabla_k R_{il} = \nabla_k (g^{il} R_{il}) = \nabla_k R.$$

The covariant derivative of $Rg_{ij} = nR_{ij}$ is $g_{ij}\nabla_k R = n\nabla_k R_{ij}$. Contracting this equality leads to $\nabla_j R = n\nabla^i R_{ij}$.

We get $(n-2)\nabla_j R = 0$. Hence when $n \neq 2$ the scalar curvature $R$ must be constant. Recall that for functions $\nabla_k f = \partial_k f$, and $R$ is a function.

Solution to Exercise 5.30.

a) If $\{Y^i\}$ are the components of a vector $Y$ in a coordinate system, then $\alpha_i Y^i = g_{ij} X^j Y^i$; thus the components of $\alpha$ are $\alpha_i = g_{ij} X^j$. Likewise, $X^j = g^{ji} \alpha_i$, and with our notations we have $\alpha^j = X^j$.

Since $\mathcal{L}_X x^i = X^j \partial_j x^i = X^i$, it follows that

$$\mathcal{L}_X \eta = (X^j \partial_j \sqrt{|g|} \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

$$+ \sum_{i=1}^n \sqrt{|g|} \, dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n.$$

We have

$$\partial_j \sqrt{|g|} = \frac{1}{2} \frac{\partial_j |g|}{\sqrt{|g|}} = \frac{1}{2} g^{ik} \partial_j g_{ik} \sqrt{|g|};$$

then $\mathcal{L}_X \eta = (\frac{1}{2} X^j g^{ik} \partial_j g_{ik} + \partial_i X^i) \eta = \nabla_i X^i \eta = -\delta \alpha \eta$. Indeed,

$$dX^i = \partial_j X^i dx^j,$$

and $\nabla_i X^i = \partial_i X^i + \Gamma^i_{ij} X^j$ with

$$\Gamma^i_{ij} = \frac{1}{2} g^{ik} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}).$$
For an alternative proof we note that $\mathcal{L}_X = i(X)d + di(X)$ on the forms. Thus $\mathcal{L}_X \eta = d[i(X)\eta] = d[X^i\eta_{i_2^{\ldots i_n}}]^{\wedge} dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n$; here $\xi$ means that $\xi$ is omitted. We obtain

$$\mathcal{L}_X \eta = d\left[ \sum_{i=1}^{n} X^i(-1)^{i-1} \sqrt{|g|} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n \right],$$

$$\mathcal{L}_X \eta = \partial_i(\sqrt{|g|}X^i)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = \nabla_i X^i \eta.$$

b) We have

$$\Delta \alpha = \delta \alpha + d\delta \alpha = \delta(dx^i \wedge dx^i) - d(\nabla^j \alpha_j)$$

and

$$d\alpha_i \wedge dx^i = \partial_j \alpha_i dx^j \wedge dx^i = \nabla_j \alpha_i dx^j \wedge dx^i,$$

since $\Gamma^k_{ji} = \Gamma^k_{ij}$ implies $\Gamma^k_{ij} dx^j \wedge dx^i = 0$. Thus

$$(\Delta \alpha)_i = -\nabla^j \nabla_j \alpha_i + \nabla^j \nabla_i \alpha_j - \nabla_i \nabla^j \alpha_j = R_{ij} X^j - \nabla^j \nabla_j \alpha_i.$$

c) $\mathcal{L}_X$ is distributive and $\mathcal{L}_X dx^i = dX^i = \partial_j X^i dx^j$. Thus

$$\mathcal{L}_X g = X^k \partial_k g_{ij} dx^i \otimes dx^j + g_{ik}(dX^i \otimes dx^k + dx^i \otimes dX^k)$$

$$= (X^k \partial_k g_{ij} + g_{ik} \partial_k X^k + g_{ik} \partial_j X^k) dx^i \otimes dx^j$$

$$= (\nabla_i \alpha_j + \nabla_j \alpha_i) dx^i \otimes dx^j.$$

The result follows.

d) If $\nabla_i \alpha_j + \nabla_j \alpha_i = 0$, by contraction we get $\delta \alpha = -\nabla^i \alpha_i = 0$. Then multiplying by $\nabla^k$ gives $\nabla^k \nabla_j \alpha_k + \nabla^k \nabla_k \alpha_j = 0$. Thus, according to b), $(\Delta \alpha)_i = R_{ij} X^j + \nabla^j \nabla_i \alpha_j = R_{ij} X^j + \nabla^j \nabla_i \alpha_j - \nabla_i \nabla^j \alpha_j = 2R_{ij} X^j$.

Conversely, $d\delta \alpha = 0$ implies

$$\int (\delta \alpha)^2 dV = \langle \delta \alpha, \delta \alpha \rangle = (\alpha, d\delta \alpha) = 0.$$

Hence $\delta \alpha = 0$, which implies $\nabla_i \nabla^j \alpha^i = R_{ij} \alpha^i$. Moreover,

$$\int (\nabla_i \alpha_j + \nabla_j \alpha_i)(\nabla^i \alpha^j + \nabla^j \alpha^i)dV = 2 \int (\nabla^i \alpha^j + \nabla^j \alpha^i) \nabla_i \alpha_j dV.$$

Then since $\nabla^i \alpha_i = 0$, and according to b), integrating by parts leads to (see 5.27)

$$\int (\nabla_i \alpha_j + \nabla_j \alpha_i)(\nabla^i \alpha^j + \nabla^j \alpha^i)dV = -2 \int \alpha^j (\nabla_i \nabla^i \alpha_j + R_{ij} \alpha^i) dV$$

$$= -2 \int \alpha^j [2R_{ij} X^i - (\Delta \alpha)_j] dV$$

$$= 0.$$
e) We have \( \frac{1}{u} (\varphi_{*+u}^g - \varphi_*^g) = \varphi_*^g \left[ \frac{1}{u} (\varphi_{*u}^g - g) \right] \rightarrow \varphi_*^g \mathcal{L}_X g \) when \( u \) tends to zero. So \( \mathcal{L}_X g = 0 \) implies \( \frac{d}{dt} (\varphi_*^g) = 0 \), and thus \( \varphi_*^g = g \).

f) We saw just above that if \( \mathcal{L}_X g = 0 \), \( \varphi_*^g \) is an isometry. Thus the covariant derivative commutes with \( \varphi_*^g \), \( \varphi_*^g \mathcal{d} = \mathcal{d} \varphi_*^g \). This implies \( \mathcal{L}_X \mathcal{d} = \mathcal{d} \mathcal{L}_X \). We have \( dw = 0 \) and \( \delta \omega = 0 \), since \( \Delta \omega = 0 \) (see 5.20). So \( \mathcal{L}_X \omega \) is homologous to zero:

\[
\mathcal{L}_X \omega = [i(X) d + d i(X)] \omega = d[i(X) \omega].
\]

On the other hand, \( \delta \mathcal{L}_X \omega = \mathcal{L}_X \delta \omega = 0 \). Hence \( \mathcal{L}_X \omega = 0 \), since \( \mathcal{L}_X \omega \) is homologous to zero (exact) and coclosed.

Solution to Problem 5.31.

a) Since \( R_{ijkl} = 0 \), according to the expression of \( R_{ijkl} \) (see 4.7) we get

\[
d\omega_{ij}^k = d\Gamma_{ij}^k \wedge dx^l = \partial_i \Gamma_{ij}^k dx^i \wedge dx^l = \frac{1}{2} (\partial_i \Gamma_{ij}^k - \partial_j \Gamma_{ij}^k) dx^i \wedge dx^l
\]

\[
= -\frac{1}{2} (\Gamma_{ij}^m \Gamma_{ml}^k - \Gamma_{ij}^m \Gamma_{ml}^k) dx^i \wedge dx^l = \Gamma_{ij}^m \Gamma_{ml}^k dx^i \wedge dx^l.
\]

Then, as \( \Gamma_{ml}^k = \Gamma_{im}^k \), it follows that \( d\omega^k_j = \omega_j^m \wedge \omega^k_m \).

b) \( \Gamma_{i\gamma} = \Gamma_{ij} B_{\beta}^i B_{\alpha}^j A_{\alpha}^i - B_{\alpha}^j \partial_\beta A_{\alpha}^i \) are the Christoffel symbols on \((\Omega, \varphi)\). Recall that \((B_{\beta}^i)\) is the inverse matrix of \((A_{\alpha}^j))\). We want that \( \Gamma_{i\gamma} = 0 \); thus \( \partial_\beta A_{\alpha}^i = \Gamma_{ij} B_{\alpha}^j A_{\alpha}^i \). Multiplying by \( A_{\beta}^j \) yields \( \partial_\alpha A_{\beta}^j = \Gamma_{ij} A_{\alpha}^i \), which is

\[
(i) \quad \{ dA_{\beta}^j = A_{\alpha}^i \omega_{ij}^k = 0 \}.
\]

c) We choose \( p = n + n^2 \); the \( A_{\alpha}^i \) and the coordinates \( x^i \) will be the variables. The system (i) is integrable since it is closed:

\[
d(A_{\alpha}^i \omega_{ij}^k) = dA_{\alpha}^i \wedge \omega_{ij}^k + A_{\alpha}^i d\omega_{ij}^k = (dA_{\alpha}^i - A_{\beta}^i \omega_{ij}^k) \wedge \omega_{ij}^k.
\]

Moreover, the system is of order \( n^2 \). This is obvious, since each \( dA_{\alpha}^i \) appears only once in the system.

Thus there are \( n^2 \) functions \( F_{\alpha}^i \) of \( A_{\alpha}^j \) and \( x^i \) which are constant. But since the system \( \{ dF_{\alpha}^i = 0 \} \) is equivalent to (i), which is of order \( n^2 \) with respect to the \( A_{\alpha}^j \), it is the same for the system \( \{ dF_{\alpha}^i = 0 \} \). Hence it may be written \( \partial x^i / \partial x^j = A_{\alpha}^j (x^i) \); that is,

\[
(ii) \quad \{ dy^\alpha = A_{\alpha}^j (x^i) dx^i \}.
\]

d) System (ii) is integrable since \( dA_{\alpha}^i (x^i) \wedge dx^j = 0 \). Indeed, we have

\[
\partial_\alpha A_{\beta}^j = A_{\alpha}^i \Gamma_{ij}^k = A_{\alpha}^i \Gamma_{ji}^k = \partial_j A_{\alpha}^i.
\]
Hence there are $n$ functions $y^\alpha(x^i)$ defined in a neighbourhood $\Omega$ of $P$ which are the coordinates corresponding to the local chart $(\Omega, \varphi)$ which we wanted to find.

**Solution to Problem 5.32.**

a) We have

$$g_{ij}(x) = \varepsilon_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} = \sum_{\alpha=1}^{p} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\alpha}{\partial x^j}.$$ 

Let $\tilde{Y}$ be a vector field on $\mathbb{R}^p$ whose restriction to $M$ is $Y$:

$$(\tilde{D}_X \tilde{Y})^\beta = X^\alpha (\partial_\alpha \tilde{Y}^\beta + \tilde{\Gamma}^\beta_{\alpha\gamma} \tilde{Y}^\gamma) = X^i (\partial_i Y^\beta + \tilde{\Gamma}^\beta_{ir} Y^r);$$

we have chosen a coordinate system $\{y^\alpha\}$ in a neighbourhood of $Q$ in $\mathbb{R}^p$ such that $y^\alpha = x^\alpha$ for $1 \leq \alpha \leq n$. $\tilde{D}_X \tilde{Y}$ depends on $\tilde{Y}$ only by $(\partial_i Y^\beta)_Q$ and $Y(Q)$. So, $\tilde{D}_X Y$ is well defined.

b) As $X \in T_Q(\mathbb{R}^p)$, we have $\tilde{D}_X \tilde{Y} \in T_Q(\mathbb{R}^p)$. Thus $\Pi_Q \tilde{D}_X Y \in T_Q(M_n).$

The restriction to $T_Q(M_n) \times \Gamma(M_n)$ of $(X,Y) \rightarrow \Pi_Q \tilde{D}_X Y$ is bilinear since $\Pi_Q$ is linear. Furthermore, if $f \in C^1$ on a neighbourhood of $Q$ in $M$, then

$$[D_X(fY)]_Q = \Pi_Q[X(fY) + f(Q)\tilde{D}_X Y] = [X(f)Y + fD_X Y]_Q.$$ 

But $\Pi_Q$ is $C^\infty$, and thus the differentiability condition holds.

Hence $(X, Y) \rightarrow D_X Y$ defines a connection with vanishing torsion tensor. Indeed, letting $\tilde{X}$ be a vector field on $\mathbb{R}^p$ whose restriction to $M$ is $X$, we have $\tilde{D}_X \tilde{Y} - \tilde{D}_Y \tilde{X} = [\tilde{X}, \tilde{Y}]$, because $(\mathbb{R}^p, \varepsilon)$ is a Riemannian manifold. Applying $\Pi_Q$ to this equality yields

$$D_X Y - D_Y X = \Pi_Q [\tilde{X}, \tilde{Y}] = [X, Y],$$

since $[\tilde{X}, \tilde{Y}]_Q = (\tilde{X}^\alpha \partial_\alpha \tilde{Y}^\beta - \tilde{Y}^\alpha \partial_\alpha \tilde{X}^\beta)_Q = X^i \partial_i Y^\beta - Y^i \partial_i X^\beta$ (see a)). In fact we have more: as $X$ and $Y$ belong to $T(M)$, $\partial_i Y^\beta = 0$ for $\beta > n$. Moreover, by definition $g(X, Y) = \varepsilon(X, Y)$. For a vector field $Z$ on $M$, applying $D_Z$ to this equality at $Q$ gives

$$(D_Z g)(X, Y) + g(D_Z X, Y) + g(X, D_Z Y)$$

$$= \varepsilon(\tilde{D}_Z \tilde{X}, \tilde{Y}) + \varepsilon(\tilde{X}, \tilde{D}_Z \tilde{Y}),$$

since $D_Z = \tilde{D}_Z$ on functions. But $\varepsilon(\tilde{D}_Z \tilde{X}, \tilde{Y}) = \varepsilon(D_Z X, Y)$, because $Y$ is orthogonal to $\tilde{D}_Z \tilde{X} - D_Z X$. Hence $D_Z g = 0$.

c) We have

$$H(X, Y) - H(Y, X) = \tilde{D}_X Y - \tilde{D}_Y X - (D_X Y - D_Y X)$$

$$= [\tilde{X}, \tilde{Y}]_Q - [X, Y]_Q = 0$$

(we saw this result in b)).
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d) Let \((\Omega_i, \varphi_i)_{i \in I}\) be an atlas for \(M\) compatible with the orientation, and \(\{x^i_j\}\) the associated coordinate systems. At each point \(Q \in M\), we choose \(N\) such that the basis \((\partial/\partial x^1_j, \cdots, \partial/\partial x^n_j, N)\) is positive in \(\mathbb{R}^{n+1}\). This definition of \(N\) makes sense; it does not depend on \(i\). As \(\mathcal{E}(Y, N) = 0\), we have \(\mathcal{E}(\tilde{D}_X Y, N) + \mathcal{E}(Y, \tilde{D}_X N) = 0\). The result follows since \(\mathcal{E}(D_X Y, N) = 0\).

Solution to Problem 5.33.

a) \(y \rightarrow d(x, y)\) is a strictly positive and continuous function on a compact set \(\gamma\). Thus there exists \(y_0 \in \gamma\) such that \(d(x, y_0) = d(x, \gamma) = a > 0\).

b) We have \(d(z, y) \geq d(z, y_0)\) for all \(y \in \gamma\), as otherwise the arc \(\widehat{zx} \cup \widehat{zy}\) would be shorter than the arc \(\widehat{zy}_0\). Thus \(d(z, y_0) = d(z, \gamma)\).

If \(b\) is small enough, \(\Sigma_b \subset \Omega\), the open set of the local chart \((\Omega, \exp^{-1})\). We know that there is a geodesic from \(z\) to \(Q\) of length \(d(z, Q)\), and furthermore there is the geodesic of the exponential map which is included in \(\Omega\). Its length is \(b = \sqrt{g_z(\tilde{X}, \tilde{X})}\). If the first geodesic is not the second, the first geodesic must go outside \(\Omega\) and its length would be greater than \(b\) (see 5.10).

c) Define \(S_b = \{P \in M \mid d(z, P) < b = d(z, y_0)\}\). Then \(\gamma \cap S_b = \emptyset\); thus \(Y = (\frac{dt}{du})_{u = u_0}\) is tangent to \(\Sigma_b\). Moreover, according to Proposition 5.13, \((\frac{dC}{dt})_{t = a}\) is perpendicular to \(\Sigma_b\). The result follows.

d) We have \(D_{dC/\partial t}[g_C(t)(e(t), \frac{dC}{dt})] = 0\) and \(D_{dC/\partial t}[g_C(t)(e(t), e(t))] = 0\), since \(Dg = 0\), \(D_{dC/\partial t} \frac{dC}{dt} = 0\) and \(D_{dC/\partial t} e(t) = 0\).

Thus both expressions are constant along \(C\), so they are equal to their values at \(y_0\).

e) The maps \(t \rightarrow \lambda \sin \frac{x t}{\lambda}, t \rightarrow C(t)\) and \(t \rightarrow e(t)\) are differentiable. Moreover, \((u, Q, \tilde{X}) \rightarrow C(u, Q, \tilde{X})\) is differentiable, so the result follows. Also, \(C_\lambda(0) = C(0, x, e(0)) = x\) and \(C_\lambda(a) = C(\lambda, y_0, Y) = \gamma(u_1)\), a point of \(\gamma\).

Since

\[
\left(\frac{dC(\lambda, y_0, Y)}{d\lambda}\right)_{\lambda=0} = \left(\frac{d\gamma}{du}\right)_{u=u_0} = Y,
\]

both geodesics are the same. Thus \(u_1 = u_0 + \lambda\).

f) We have

\[
f(\lambda) = \int_0^a \sqrt{g_{C_\lambda(t)} \left(\frac{dC_\lambda}{dt}, \frac{dC_\lambda}{dt}\right)} dt.
\]

On \(C (\lambda = 0)\) we know that the square root is constant, so it is equal to 1 since \(f(0) = a\). We can differentiate under the integral sign with
respect to \( \lambda \). Moreover, the first derivative of \( g_{ij} \) on \( C \) is zero. Hence

\[
f'(0) = \int_0^a g_{C(t)} \left( v(t), \frac{dC}{dt} \right) dt
\]

with \( v(t) = \frac{d}{dt} \left( \frac{\partial}{\partial \lambda} C(\lambda(t)) \right)_{\lambda=0} \), since we can permute the derivatives (the function is \( C^2 \)). Now

\[
\left( \frac{\partial C_\lambda(t)}{\partial \lambda} \right)_{\lambda=0} = \sin \frac{\pi t}{2a} e(t)
\]

(see 5.10). Hence

\[
v(t) = \frac{\pi}{2a} \cos \frac{\pi t}{2a} e(t) + \sin \frac{\pi t}{2a} \frac{D}{dt} e(t) = \frac{\pi}{2a} \cos \frac{\pi t}{2a} e(t).
\]

The integral vanishes since \( e(t) \) is orthogonal to \( \frac{dC}{dt} \). It is obvious that \( f'(0) = 0 \), according to the definition of \( y_0 \).

g) Since \( f(\lambda) \) is minimum at \( \lambda = 0 \), we have

\[
\left( \frac{d^2 f(\lambda)}{d\lambda^2} \right)_{\lambda=0} \geq 0.
\]

Now

\[
f''(0) = \frac{1}{2} \int_0^a \partial_{kl} g_{ij}(C(t)) e^k(t) e^l(t) \frac{dC^i}{dt} \frac{dC^j}{dt} \sin^2 \frac{\pi t}{2a} dt
\]

\[
+ \left( \frac{\pi}{2a} \right)^2 \int_0^a \cos^2 \frac{\pi t}{2a} g_{ij}(C(t)) e^i(t) e^j(t) dt.
\]

But \( R_{ikjl} = -\frac{1}{2} \partial_{kl} g_{ij} \) on \( C \), and, moreover,

\[
\int_0^a \cos^2 \frac{\pi t}{2a} dt = \int_0^a \sin^2 \frac{\pi t}{2a} dt = \frac{a}{2}.
\]

Thus

\[
0 \leq f''(0) \leq \frac{a}{2} \left[ \left( \frac{\pi}{2a} \right)^2 - k^2 \right] g_{w_0}(Y, Y)
\]

and \( a \leq \pi/2k \).
Solution to Exercise 5.34.

a) The parameter $s$ of arc length of $C$ is proportional to $t$ (see 5.10). Without loss of generality, we can suppose that $t = s$. Set $x_P = C(s_P)$. We have $d(x_P, x_Q) \leq |s_P - s_Q|$. Hence $\{x_P\}$ is a Cauchy sequence converging to a point, say $Q \in M$, which does not depend on the sequence $\{s_P\}$.

b) Let $i_0$ be such that $C(t_{i_0}) \in \Omega$ with $t_{i_0} > \alpha - r$. Since there is a unique geodesic included in it from $Q$ to $C(t_{i_0})$ (Theorem 5.11), it follows that $C(t)$ for $t \geq t_{i_0}$ belongs to the geodesic from $Q$ to $C(t_{i_0})$. According to 5.10 the geodesic can be extended for all values of $t$ such that $\alpha \leq t < \alpha + r$.

c) Consequently all geodesics are infinitely extendable.

Solution to Exercise 5.35.

Let $\psi$ be a harmonic differential $p$-form. Then $d\psi = 0$ and $\delta\psi = 0$ according to Proposition 5.20.

a) If $\Delta \alpha = \gamma$, then

$$\langle \gamma, \psi \rangle = \langle d\delta\alpha, \psi \rangle + \langle \delta d\alpha, \psi \rangle = \langle \delta\alpha, \delta\psi \rangle + \langle d\alpha, d\psi \rangle = 0.$$ 

So $\gamma$ is orthogonal to $H_p$. Now let us suppose that $\langle \gamma, \psi \rangle = 0$ for any harmonic differential $p$-form $\psi$. Applying de Rham's Theorem 5.22 three times, we see that there exist two differential forms $\beta$ and $\omega$ such that $\gamma = d\delta\beta + \delta d\omega$. Setting $\alpha = d\beta + \delta\omega$, we get $\Delta \alpha = \gamma$.

b) According to (a) we only have to prove that $\alpha$ is unique in $A_p$. If $\Delta \alpha = 0$ then $\alpha \in H_p$; thus $\alpha = 0$ since $\alpha \in A_p$. $G$ exists, $\alpha = G\gamma$, and $\gamma = \Delta \alpha = \Delta G\gamma$ implies $\Delta G = \text{Identity on } A_p$. Let $\alpha \in A_p$; then $\Delta \alpha = d\delta\alpha + \delta d\alpha$, and according to (a) $\alpha = G\Delta \alpha$, $G\Delta = \text{Identity on } A_p$.

c) Let $A = \bigcup_{p=0}^\infty A_p$. We suppose that a linear operator $T$ on $A$ satisfies $T \circ \Delta = \Delta \circ T$. If $\alpha \in A$, there exists $\beta \in A$ such that $\alpha = G\beta$; thus $GT\beta = GT \circ \Delta G\beta = G\Delta \circ TG\beta = TG\beta$ since $\Delta G = G\Delta = \text{Identity on } A$. As $d, \delta$ and $\ast$ are $T$ operators, the result follows.

d) We have $\gamma' = \gamma + d\alpha$ and $\phi' = \phi + d\beta$ for some $\alpha$ and $\beta$. Moreover,

$$\int (\gamma + d\alpha) \wedge (\phi + d\beta) = \int \gamma \wedge \phi.$$

Indeed, since

$$(d\alpha) \wedge (\phi + d\beta) + \gamma \wedge d\beta = d[\alpha \wedge (\phi + d\beta)] + (-1)^p d(\gamma \wedge \beta),$$

according to Stokes' formula the integral on the right hand side vanishes.
e) Since $\ast \varphi$ is harmonic, $\ast \varphi$ is closed. Thus $\langle \varphi, \ast \varphi \rangle$ makes sense. Also,
$\langle \varphi, \ast \varphi \rangle = \int \varphi \wedge \ast \varphi = \int (\varphi, \varphi) \eta = 0$ implies $\varphi \equiv 0$. If $\varphi$ is not zero, then $\langle \varphi, \ast \varphi \rangle \neq 0$.

**Solution to Exercise 5.36.**

a) Let $Q$ be the point opposite to $P$ on $S_2$. Then $\Omega = S_2 - \{Q\}$ is in bijection with $B \in \mathbb{R}^2$, where $(\theta, r)$ is a polar coordinate system, $B$ being the open disk of radius $\pi$ centered at $0 \in \mathbb{R}^2$. We write

$$\begin{cases} x = \sin r \cos \theta, \\ y = \sin r \sin \theta, \\ z = \cos r. \end{cases}$$

We have $\frac{\partial x}{\partial \theta} = -\sin r \sin \theta$, and $\frac{\partial y}{\partial \theta} = \sin r \cos \theta$. Thus

$$g_{\theta \theta} = \left( \frac{\partial x}{\partial \theta} \right)^2 + \left( \frac{\partial y}{\partial \theta} \right)^2 + \left( \frac{\partial z}{\partial \theta} \right)^2 = \sin^2 r.$$

Likewise we compute $g_{\theta r}$ and $g_{rr}$:

$$g_{\theta r} = \left( \frac{\partial x}{\partial \theta} \right) \left( \frac{\partial x}{\partial r} \right) + \left( \frac{\partial y}{\partial \theta} \right) \left( \frac{\partial y}{\partial r} \right) + \left( \frac{\partial z}{\partial \theta} \right) \left( \frac{\partial z}{\partial r} \right)$$

$$= \sin r [\cos r \cos \theta(- \sin \theta) + \cos r \sin \theta \cos \theta]$$

$$= 0,$$

$$g_{rr} = \left( \frac{\partial x}{\partial r} \right)^2 + \left( \frac{\partial y}{\partial r} \right)^2 + \left( \frac{\partial z}{\partial r} \right)^2$$

$$= \cos^2 r + \sin^2 r = 1.$$

b) $\partial_r g_{\theta \theta} = \sin 2r$, and the other first derivatives of the components of the metric tensor vanish. Thus $\Gamma^r_{\theta \theta} = -\frac{1}{2} \partial_r g_{\theta \theta} = -\frac{1}{2} \sin 2r$ and

$$\Gamma^\theta_{r \theta} = \frac{1}{2 \sin^2 r} \partial_r g_{\theta \theta} = \cotan r.$$

Moreover, $g^{rr} = 1$, $g^{\theta r} = 0$ and $g^{\theta \theta} = \sin^{-2} r$ ($r \neq 0$).

We know that there is only one component to compute for the curvature tensor:

$$R^r_{\theta r \theta} = \partial_r \Gamma^r_{\theta \theta} - \Gamma^r_{\theta \theta} \Gamma^\theta_{r \theta} = -\cos 2r + \sin r \cos r \cotan r = \sin^2 r.$$

We infer from this value that

$$R_{\theta \theta} = \sin^2 r, \quad R_{\theta r} = 0, \quad R_{rr} = R_{r \theta r \theta} g^{\theta \theta} = 1.$$

The scalar curvature $R = 2$, and we have $R_{ij} = g_{ij}$.

C) According to the geodesic equation, the curve $t \rightarrow [\theta(t), r(t)]$ is a geodesic if

$$\theta'' + \Gamma^\theta_{r \theta} r' \theta' = 0 \quad \text{and} \quad r'' + \Gamma^r_{\theta \theta} (\theta')^2 = 0.$$
The curve $\theta = \text{Constant}$ is a geodesic for any value of the constant.

The curve $r = \text{Constant}$ is a geodesic if $\Gamma^r_{\theta \theta} = 0$, that is, for $r = \pi/2$. In this case, $\theta'' = 0$. Thus $\theta = at + b$.

The meridians and the equator are geodesics. On the sphere, only the great circles are geodesics; that is, the intersections of $S_n$ with the planes through the center of $S_n \subset \mathbb{R}^{n+1}$.

d) Since $g^{r \theta} = 0$, when $f$ depends only on $r$ we have

$$-\Delta f = g^{ij} \nabla_i \nabla_j f = \nabla_{rr} f + g^{\theta \theta} \nabla_{\theta \theta} f = \nabla_{rr} f - g^{\theta \theta} \Gamma^r_{\theta \theta} \partial_r f$$

and

$$\Delta \cos r = \cos r + \frac{1}{\sin^2 r} (-\sin r \cos r)(-\sin r) = 2 \cos r.$$  

The corresponding eigenvalue is $\lambda = 2$, and it is the first nonzero eigenvalue according to Lichnerowicz's Theorem 5.27. Here, with $k = 1$ and $n = 2$, we find that $\lambda_1 \geq 2$. Indeed, the Ricci curvature of the sphere $S_2$ is 1 ($g_{ij} \xi^i \xi^j = 1$ implies $R_{ij} \xi^i \xi^j = 1$).

**Solution to Exercise 5.37.**

a) We have

$$\Gamma^i_{jk} = \frac{1}{2} g^{ij}(\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik}) = \frac{1}{2} g^{ij} \partial_k g_{ij}.$$  

Moreover, $\partial_k |g| = |g| g^{ij} \partial_k g_{ij}$, and hence

$$\Gamma^i_{jk} = \frac{1}{2} \partial_k \log |g| = \partial_k \log \sqrt{|g|}.$$  

b) We have

$$-\Delta \varphi = g^{ij} \nabla_i \nabla_j \varphi = \nabla_i (g^{ij} \nabla_j \varphi) = \partial_i (g^{ij} \partial_j \varphi) + \Gamma^i_{jk} g^{kj} \partial_j \varphi,$$

since $\nabla g = 0$.

If $\varphi(Q) = f(r)$, then, in a geodesic coordinate system, $\partial_j \varphi = 0$ when $j \neq r$. Since $g^{r \theta} = 0$ and $g^{rr} = 1$,

$$-\Delta \varphi = f'' + f' \partial_r \log \sqrt{|g|}.$$  

**Solution to Problem 5.38.**

a) $\Psi : x \rightarrow y$ is a bicontinuous bijection of $\Omega$ on $\pi$. $(\Omega, \Psi)$ is a local chart $C$, and the corresponding coordinates are $\{y^k\}$.

b) We have $x^{n+1} = -\cos \alpha$, $\sum_{i=1}^n (x^i)^2 = \sin^2 \alpha$ and $\sum_{j=1}^n (y^j)^2 = \rho^2$.

Since $x^i = ky^i$ ($1 \leq i \leq n$) for some real number $k > 0$, it follows that
\[ \sin^2 \alpha = k^2 \rho^2. \] Thus \( k = \frac{\sin \alpha}{\rho}. \) Considering the two triangles \( P O y \) and \( P x Q \) yields

\[ \rho = \frac{2 \sin \frac{\alpha}{2}}{2 \sin \frac{\pi - \alpha}{2}} = \tan \frac{\alpha}{2}. \]

Hence

\[ x^i = \frac{\sin \alpha}{\rho} y^i = \frac{2}{1 + \rho^2} y^i \]

for \( 1 \leq i \leq n, \) and

\[ x^{n+1} = -\frac{1 - \rho^2}{1 + \rho^2} = 1 - \frac{2}{1 + \rho^2}. \]

c) We have

\[ g_{ij} = \frac{\partial \Phi^\mu}{\partial y^i} \frac{\partial \Phi^\nu}{\partial y^j} E_{\mu \nu}. \]

Choose polar coordinates on \( \pi, \) and cylindrical coordinates on \( \mathbb{R}^{n+1} \) (those of \( \pi \) and \( x^{n+1} \)). In local coordinates \( \theta^j \) on \( S_{n-1} \) we have \( \theta^j(x) = \theta^j(y) \) \((1 \leq j \leq n-1)\) and

\[ \sum_{i=1}^{n} (x^i)^2 = \sin^2 \alpha = \left( \frac{2 \rho}{1 + \rho^2} \right)^2 \]

We have

\[ g_{\rho \rho} = \left( \frac{\partial \sin \alpha}{\partial \rho} \right)^2 + \left( \frac{\partial x^{n+1}}{\partial \rho} \right)^2 \]

\[ = (\cos^2 \alpha + \sin^2 \alpha) \left( \frac{\partial \alpha}{\partial \rho} \right)^2 = \frac{4}{(1 + \rho^2)^2} \]

and

\[ g_{\theta^i \theta^j} = E_{\theta^i \theta^j} = \delta_i^j \sin^2 \alpha = \frac{4 \rho^2}{(1 + \rho^2)^2} \delta_i^j. \]

d) We verify that \( g = (4/(1 + \rho^2)^2) E_\pi, \) so \( f(\rho) = 4(1 + \rho^2)^{-2}. \)

e) In polar coordinates the straight lines have the equations \( \theta^j = \text{Constant} \) and \( \rho = h(s). \) They will be geodesics for the metric \( g \) if \( \Gamma_{\rho \rho}^\theta = 0, \) \( 1 \leq i \leq n - 1, \) and \( h''(s) + \Gamma_{\rho \rho}(s)[h'(s)]^2 = 0. \) We easily verify that \( \Gamma_{\rho \rho}^\theta = 0. \) The arc length satisfies

\[ ds = \sqrt{g_{\rho \rho}} d\rho = \frac{2d\rho}{1 + \rho^2}, \]

and so \( s = 2 \arctan \rho, \rho = \tan \frac{s}{2}, \rho' = \frac{1}{2}(1 + \rho^2), \) and \( \rho'' = \rho \rho'. \) Moreover,

\[ \Gamma_{\rho \rho}^\rho = \frac{1}{2} \partial_\rho \log g_{\rho \rho} = -\frac{2 \rho}{1 + \rho^2}. \]
Thus

\[ h'' + \Gamma_{\rho\rho}^\rho(h')^2 = \frac{\rho(1 + \rho^2)}{2} - \frac{2\rho}{1 + \rho^2} \left( \frac{1 + \rho^2}{2} \right)^2 = 0. \]

f) We found all the geodesics of \( S_n \) through \( Q \). So the geodesics of the sphere \( S_n \) are the great circles, and only them.

g) \( r = \alpha = 2 \arctan \rho \).

h) Of course \( g_{\rho\rho} = 1, g_{\rho\theta} = 0 \), and by symmetry \( g_{\theta\theta} \) depends only on \( r \). Now the length of the horizontal circle through \( x \) has two expressions:

\[ 2\pi \sqrt{g_{\theta\theta}} \quad \text{and} \quad 2\pi \sin \alpha. \]

Thus \( g_{\theta\theta} = \sin^2 r \).

i) We have \( \varphi(x) = -\cos r \).

j) We have \(-\Delta \varphi = \cos r + (n - 1) \sin r \frac{\cos r}{\sin r} = n \cos r \). Also, \( \Delta \varphi = n \varphi \). So \( \varphi \) is an eigenfunction for \( \Delta \) on \( (S, g) \), and \( \lambda_1 = n \).

k) Set \( \psi = \cos^2 r + k = h(r) \). Then \( h' = -\sin 2r \) and

\[ -\Delta \psi = -2 \cos 2r - \sin 2r \frac{(n - 1) \cos r}{\sin r} = 2 - 2(n + 1) \cos^2 r \]

according to the formula of Exercise 5.37.

We want that \( \Delta \psi = 2(n + 1) \psi \). This implies \( \lambda_2 = 2(n + 1) \) and \( (n + 1)k = p - 1 \), \( k = -1/(n + 1) \).

Solution to Problem 5.39.

a) According to the Hodge decomposition theorem, there exist a real number \( a \) and a differential \( (n - 1) \)-form \( \varphi \) such that \( \omega = a\eta + d\varphi \), \( \eta \) being a nonzero harmonic \( n \)-form (all harmonic \( n \)-forms are proportional). We have

\[ \int_M f^* \omega = a \int_M f^* \eta + \int_M d(f^* \varphi) = a \int_M f^* \eta \]

according to Stokes' formula, since \( f^* \) and \( d \) commute. Thus \( k \) does not depend on \( \omega \). Since \( \int_W \omega = a \int_W \eta \), it follows that

\[ k = \int_M f^* \eta / \int_W \eta. \]

Indeed, \( \int_W \eta \neq 0 \), since there are \( n \)-forms \( \omega \) whose integral is not zero. For instance, if \( (\Omega, \varphi) \) is a local chart with coordinates \( \{x^i\} \), choose \( \omega = h(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \) with \( h \) smooth, \( h \geq 0 \) and \( \text{supp } h \subset \Omega \).

b) Since \( f \) is continuous, \( f(M) \) is a compact set of \( W \). Thus \( \theta = W \setminus f(M) \) is an open set, not empty if \( f \) is not onto. Consider on \( \Omega \subset \theta \), as above, \( \omega = h(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \). Then \( \int_W \omega \neq 0 \) and \( \int_M f^* \omega = 0 \). Thus \( k = 0 \) in this case.
c) Let $P_i \in f^{-1}(Q)$. Since $(df)_{P_i}$ is of rank $n$, $f$ is a diffeomorphism on a neighbourhood $\hat{\Omega}_i$ of $P_i$. The proof is by contradiction. Suppose $f^{-1}(Q)$ is not finite. There exists a sequence $\{P_j\} \subset f^{-1}(Q)$ such that $P_j \to P$, which implies by continuity that $f(P) = Q$. So $f$ would be a diffeomorphism on a neighbourhood $\hat{\Omega}$ of $P$. This is impossible, since for large $j$ we would have $P_j \in \hat{\Omega}$. Hence $f^{-1}(Q) = \{P_1, P_2, \ldots, P_m\}$. Choose the $\hat{\Omega}_i$ disjoint, and set $\theta = \bigcap_{i=1}^{m} f(\hat{\Omega}_i)$ and $\Omega_i = \hat{\Omega}_i \cap f^{-1}(\theta)$. Then $f^{-1}(\theta) = \bigcup_{i=1}^{m} \Omega_i$.

d) $f|\Omega_i$ is a diffeomorphism; thus $\int_W \omega = \pm \int_{\Omega_i} f^{*}\omega$ according to the chosen orientations on $M$ and $W$. Hence $k$ is an integer ($|k| \leq m$).

e) Let $\vec{x}$ be a unit vector. Then $\mathcal{E}(\vec{x}, X(x)) = 0$, since $X(x) \in T_x(S_n)$. Thus

$$\sum_{j=1}^{n+1} (y^j)^2 = \|\cos \pi t \vec{x} + \sin \pi t X\|^2 = \cos^2 \pi t + \sin^2 \pi t = 1.$$ 

So $F_t$ is a differentiable map of $S_n$ into $S_n$. Let $\eta$ be the oriented volume $n$-form on $S_n$. Since the map

$$\mathbb{R} \times S_n \ni (t, x) \to F_t(x) \in S_n$$

is $C^1$, it follows that $t \to \int_{S_n} F^*_t \eta$ is continuous. Thus $t \to k(t)$ is also continuous. Then $k(t)$ is constant, since the $k(t)$ are integers.

f) $F_0(x) = x$ and $F_0 = \text{Id}$; hence $k(0) = 1$.

Also, $F_1(x) = -x$, and $F_1$ is the restriction to $S_n$ of $-\text{Id}$ on $\mathbb{R}^{n+1}$. Set $\tilde{F} = -\text{Id}$ on $\mathbb{R}^{n+1}$. If $\tilde{\eta}$ is the oriented volume $(n+1)$-form on $\mathbb{R}^{n+1}$, the orientation of $S_n$ is the natural orientation induced by that of $\mathbb{R}^{n+1}$. Also, $\tilde{F}^* \tilde{\eta} = (-1)^{n+1} \tilde{\eta}$. Let $\tilde{\nu}$ the unit normal vector to $S_n$ oriented to the outside; $\tilde{\nu}(x)$ has $\{x^j\}$ for components. Moreover, $\eta = i(\tilde{\nu}) \tilde{\eta}$ and $\tilde{F}_* \tilde{\nu} = \nu \circ \tilde{F}$. Hence $\tilde{F}^* \eta = i(\nu) \tilde{F}^* \tilde{\eta} = (-1)^{n+1} \eta$ and $k(1) = (-1)^{n+1}$. When $n$ is even, $k(0)$ is not equal to $k(1)$. This is in contradiction with $k(t) = \text{Constant}$. So, on $S_n$ when $n$ is even, there does not exist a vector field $X$ which is nowhere zero. We would consider $X/\|X\|$.

**Solution to Exercise 5.40.**

Let $(\Omega, \varphi)$ be a normal coordinate system at $P$. For any $\varepsilon > 0$ there exists $\rho$ such that if $\|Z\| < \rho (Z \in B(\rho)$, the ball of radius $\rho$ in $\mathbb{R}^n$ centered at 0), then

$$(1 - \varepsilon)\|\xi\|^2 \leq [g_{ij}(\exp P Z)](\varphi^{-1}_* \xi)(\varphi^{-1}_* \xi)^j \leq \|\xi\|^2(1 + \varepsilon)$$

for any vector $\xi \in \mathbb{R}^n$, since $g_{ij}(P) = \delta_{ij}^j$ and $(\varphi^{-1}_*) P = \text{Id}$. If $\|tX\| < \rho/2$ and $\|tY\| < \rho/2$, an arc from $Q = \exp tX$ to $\tilde{Q} = \exp tY$ is included in $\Theta = \exp_P B(\rho)$ when its length is smaller than $\rho$. 

Since \( d(Q, \tilde{Q}) \leq d(Q, P) + d(P, \tilde{Q}) < \rho \), all differentiable curves \( \gamma \) from \( Q \) to \( \tilde{Q} \) that we must consider for the definition of \( d(Q, \tilde{Q}) \) are included in \( \Theta \), and so

\[
d(Q, \tilde{Q}) \leq L(\gamma) \leq \sqrt{1 + \varepsilon L_\epsilon(\varphi \circ \gamma)},
\]

where \( L_\epsilon \) is the Euclidean length. This is true for any \( \gamma \). Choosing \( \gamma = \varphi^{-1}([tX, tY]), [tX, tY] \) being the segment from \( tX \) to \( tY \), we find that

\[
d(\exp_P tX, \exp_P tY) \leq t\sqrt{1 + \varepsilon\|X - Y\|}.
\]

**Solution to Problem 5.41.**

a) We have

\[
D_{\frac{\partial}{\partial t}} [g_{C(t)}(e_i(t), e_j(t))] = (D_{\frac{\partial}{\partial t}} g)_{C(t)}(e_i(t), e_j(t))
+ g_{C(t)}[D_{\frac{\partial}{\partial t}} e_i(t), e_j(t)]
+ g_{C(t)}[e_i(t), D_{\frac{\partial}{\partial t}} e_j(t)].
\]

In the right hand side the three terms vanish, the first since \( Dg = 0 \) and the others since \( DdC/dt e_j(t) = 0 \) for \( 1 \leq j \leq n \). We infer that for all \( i, j \),

\[
g_{C(t)}(e_i(t), e_j(t)) = gp(e_i, e_j) = \delta_i^j.
\]

Note that \( e_1(t) = \frac{dC}{dt} \), since \( C(t) \) is a geodesic.

b) We have

\[
g_{ij}(C(t)) = \frac{\partial}{\partial x^i} \left( \frac{\partial}{\partial x^j} \right) = \delta_i^j.
\]

Moreover,

\[
[D_{\frac{\partial}{\partial t}} e_j(t)]^k = \frac{de_j^k(t)}{dt} + \Gamma_{ij}^k(C(t))e_j^i(t) = 0.
\]

Thus \( \Gamma_{ij}^k(C(t)) = 0 \) for all pairs \( j, k \), since \( e_j^k(t) = \delta_j^k \). We know nothing about the other Christoffel symbols.

c) According to the properties of the exponential mapping, \( f \) is \( C^\infty \). Also, \( f(t, 0) = C(t) \), and thus

\[
\left( \frac{\partial f}{\partial t} \right)_{(t,0)} = \left( \frac{dC}{dt} \right)_{(t,0)} = (1, 0, 0, \ldots, 0).
\]

Moreover, for \( \xi \in \mathbb{R}^{n-1} \) (\( \xi = \sum_{i=2}^{n} \xi^i e_i(t) \)) we have

\[
\left( \frac{\partial f(t, u\xi)}{\partial u} \right)_{u=0} = \sum_{i=2}^{n} \xi^i e_i(t).
\]
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Thus \((Df)_{(t,0)}\) is invertible, and \((Df)^{-1}_{(t,0)}\) from \(T_{C(t)}(M)\) to \(T_{0}(\mathbb{R}^{n})\) is

\[
\sum_{i=1}^{n} \xi^{i}e_{i}(t) \rightarrow \sum_{i=1}^{n} \xi^{i}\tilde{e}_{i}.
\]

d) Let \(d(Q, R)\) be a continuous function on the compact set \(C\). Thus the inf is achieved at least at one point \(\bar{Q}\).

e) The proof is by contradiction. We suppose that \(X = \left(\frac{d\gamma(u)}{du}\right)_{Q}\) and \(Z = \left(\frac{dC(t)}{dt}\right)_{Q}\) are not orthogonal vectors. We choose

\[
Y = Zg_{Q}(X, Z)/g_{Q}(Z, Z),
\]

for instance. We have

\[
g_{Q}(X - Y, X - Y) = g_{Q}(X, X) - [g_{Q}(X, Z)]^{2}/g_{Q}(Z, Z) < g_{Q}(X, X).
\]

Using the result of Exercise 5.40 yields, for small \(t\),

\[
b = d(\exp_{\bar{Q}} t\tilde{X}, \exp_{\bar{Q}} t\tilde{Y}) \leq (1 + \epsilon)t\|\tilde{X} - \tilde{Y}\| < t\|\tilde{X}\|
\]

with \(\tilde{X} = D\exp_{\bar{Q}}^{-1} X\) and \(\tilde{Y} = D\exp_{\bar{Q}}^{-1} Y\). Recall that \(g_{\bar{Q}}(\bar{Q}) = \delta_{i}^{j}\).

Thus the curve formed by the two arcs of geodesics, a piece of \(\gamma\) from \(Q\) to \(\exp_{\bar{Q}} t\tilde{X}\), has a shorter length than the length of \(\gamma\). But this is impossible.

f) The proof is by contradiction. Let \(\{Q_{k}\}\) be a sequence of points such that there exist \(\bar{Q}_{k_{1}}\) and \(\bar{Q}_{k_{2}}\), \(\bar{Q}_{k_{1}} \neq \bar{Q}_{k_{2}}\), with

\[
d(Q_{k}, \bar{Q}_{k_{1}}) = d(Q_{k}, \bar{Q}_{k_{2}}) = d(Q_{k}, C) < 1/k.
\]

Since \(C\) is a compact set, we can suppose that \(\bar{Q}_{k_{1}}\) and \(\bar{Q}_{k_{2}}\) tend to a point \(R\) of \(C\). Thus \(Q_{k}\) tends to \(R\) also, and, according to c), for \(k\) large

\[
Q_{k} = f(t_{k}, \bar{u}_{k}) = f(\bar{t}_{k}, \tilde{u}_{k})
\]

if \(\bar{Q}_{k_{1}} = f(t_{k}, 0)\) and \(\bar{Q}_{k_{2}} = f(\bar{t}_{k}, 0)\).

Now in a neighborhood of \(R\), according to the inverse function theorem, a point such as \(Q_{k}\), for large \(k\), is the image by \(f\) of a unique pair \((t, \bar{u})\). Thus we get the contradiction and the existence of \(\epsilon > 0\).

g) \(f\) is bijective on \([0, a[ \times B_{c}\). Moreover, it admits an inverse function \(f^{-1}\), which is differentiable according to the inverse function theorem. Thus \(f\) is a diffeomorphism of \([0, a[ \times B_{c}\) onto \(\theta\), and \((\theta, f^{-1})\) is a local chart.
h) We saw that $D_{dC(t)/dt}e_i(t) = 0$ implies $\Gamma_{ij}^k(C(t)) = 0$, $1 \leq i, j \leq n$.
Moreover let us write that $u \rightarrow f(t, u\xi)$ satisfies the geodesic equation:

$$\frac{\partial^2 f^i(t, u\xi)}{\partial u^2} + \Gamma^i_{jk}(t, u\xi) \frac{\partial f^j(t, u\xi)}{\partial u} \frac{\partial f^k(t, u\xi)}{\partial u} = 0.$$

Now $f^1(t, u\xi) = t$, and $f^i(t, u\xi) = u\xi^i$ for $i > 1$. Thus at $u = 0$ the equation above yields $\Gamma^k_{ij}(C(t))\xi^i\xi^j = 0$, for all $\xi \in \mathbb{R}^{n-1}$ and all $1 \leq k \leq n$. This implies $\Gamma^k_{ij}(C(t)) = 0$ when $2 \leq i, j \leq n$ and $1 \leq k \leq n$. So the Christoffel symbols vanish on the geodesic $C$.

**Solution to Problem 5.42.**

a) The rank of $\psi$ is 2. Indeed,

$$D\psi = \begin{pmatrix}
-(a + b \cos v) \sin u & -b \sin v \cos u \\
(a + b \cos v) \cos u & -b \sin v \sin u \\
0 & b \cos v
\end{pmatrix}.$$  

The $2 \times 2$ determinants of $D\psi$ are

$$D_1 = (a + b \cos v)b \sin v \neq 0 \quad \text{if } v \neq h\pi \ (h \in \mathbb{Z});$$

when $v = h\pi$

$$D_2 = -b(a + b \cos v) \sin u \cos v \neq 0 \quad \text{if } u \neq k\pi \ (k \in \mathbb{Z});$$

and when $u = k\pi$ and $v = h\pi$

$$D_3 = (a + b \cos v)b \cos v \cos u \neq 0.$$  

b) If $(u, v) \sim (\tilde{u}, \tilde{v})$, then $\psi(u, v) = \psi(\tilde{u}, \tilde{v})$. Thus we can define $\tilde{\psi}$ from $\psi$. Also, $\tilde{\psi}$ is differentiable of rank 2 like $\psi$. Moreover, $\tilde{\psi}$ is injective. Indeed, suppose $\psi(u, v) = \psi(\tilde{u}, \tilde{v})$. Then the equality

$$x^2 + y^2 = (a + b \cos v)^2 = (a + b \cos \tilde{v})^2$$

implies $a + b \cos v = a + b \cos \tilde{v}$, since $a > b$. Then $\cos v = \cos \tilde{v}$ and $\sin v = \sin \tilde{v}$ imply $\tilde{v} = v + 2h\pi \ (h \in \mathbb{Z})$. Thus $\tilde{u} = u + 2k\pi \ (k \in \mathbb{Z})$. Finally, $\tilde{\psi}$ is an imbedding since $\tilde{\psi}$ is proper, $C \times C$ being a compact set.

c) We have $x^2 + y^2 = a^2 + 2ab \cos v + b^2(1 - \sin^2 v)$. On $M = \psi(\mathbb{R}^2)$

$$P(x, y, z) \equiv (x^2 + y^2 - a^2 - b^2 + z^2)^2 - 4a^2(b^2 - z^2) = 0.$$  

We can write $P$ in the form

$$P(x, y, z) \equiv (x^2 + y^2 + z^2 + a^2 - b^2)^2 - 4a^2(x^2 + y^2).$$

If $P(x, y, z) = 0$, then $r^2 + z^2 + a^2 - b^2 = 2ar$ with $x = r \cos u$ and $y = r \sin u$, since $a > b$. But $(r - a)^2 = b^2 - z^2$ implies $z = b \sin v$ and $r = a + b \cos v$. Thus $M = P^{-1}(0)$. 
d) $DP$ is of rank 1 on $M$. Indeed, if $Q = x^2 + y^2 + z^2 - a^2 - b^2$, then $DP = (4xQ, 4yQ, 4zQ + 8a^2z)$. Since $Q = 2abc \cos v$, $Q = 0$ implies $z^2 = b^2$, and then $\frac{\partial P}{\partial z} = 8a^2z \neq 0$. If $Q \neq 0$ and $z \neq 0$, then $\frac{\partial P}{\partial z} \neq 0$. As $(0, 0, 0) \notin M$ since $P(0, 0, 0) = (a^2 - b^2)^2 > 0$, it follows that $DP$ is of rank 1 on $M$, and $M$ is a submanifold of $\mathbb{R}^3$. $\psi$ is differentiable and injective. To $\psi$ there corresponds $\Phi: C \times C \to M$, which is differentiable and bijective. Since $D\Phi$ is of rank 2, $\Phi$ is locally invertible and its inverse $D\Phi^{-1}$ is differentiable. $M$ is diffeomorphic to $C \times C$.

e) The components of the metric $g$ in the coordinate system $(u, v)$ are

$$
\begin{align*}
g_{uu} &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = (a + b \cos v)^2, \\
g_{uv} &= \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} = 0, \\
g_{vv} &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = b^2.
\end{align*}
$$

The Christoffel symbols that may be nonzero are those in which $\partial_v g_{uu}$ appears (those with one $v$ and two $u$'s):

$$
\Gamma^v_{uu} = \frac{(a + b \cos v)}{b} \sin v \quad \text{and} \quad \Gamma^u_{vv} = -\frac{b \sin v}{a + b \cos v}.
$$

f) We have

$$
R = 2g^{vv} R^u_{vvv} = \frac{2\cos v}{b(a + b \cos v)},
$$

because (see 4.7)

$$
R^u_{vvv} = \frac{b \cos v}{a + b \cos v} + \frac{b^2 \sin^2 v}{(a + b \cos v)^2} - \frac{b^2 \sin^2 v}{(a + b \cos v)^2} = \frac{b \cos v}{a + b \cos v}.
$$

Since $\sqrt{|g|} = b(a + b \cos v)$,

$$
\int_M R \, dV = 4\pi \int_0^{2\pi} \cos v \, dv = 0.
$$

g) The differential equations satisfied by the arcs of geodesics are

$$
\frac{d^2 u}{dt^2} = -\left(\Gamma^v_{uu} + \Gamma^u_{vu}\right) \frac{du}{dt} \frac{dv}{dt} = 2\frac{b \sin v}{a + b \cos v} \frac{du}{dt} \frac{dv}{dt}
$$

and

$$
\frac{d^2 v}{dt^2} = -\Gamma^v_{uu} \left(\frac{du}{dt}\right)^2 = -\frac{(a + b \cos v)}{b} \sin v \left(\frac{du}{dt}\right)^2.
$$
The equation of a plane through Oz is \( u = u_0 \). This implies \( \frac{du}{dt} = 0 \), and the equations have a solution \( u = u_0, v = v_0 + \mu t \). The intersections of planes through Oz with \( M \) are geodesics.

The equation of a plane orthogonal to Oz is \( z = z_0 \); thus \( v = v_0 \). Then \( u = u_0 + \lambda t \) is a solution of the first equation, but the second then implies \( \sin v = 0 \). Hence \( v = k\pi \) \((k \in \mathbb{Z})\). The geodesics which are in a plane orthogonal to Oz are those which are in the plane \( z = 0 \).

h) If \( \frac{dv}{dt} = 0 \) at \((u_0, v_0)\), then \( u(t) = u_0 \) and \( v(t) = v_0 + \lambda t \) is a solution of the equations. When initial conditions are given, the solution is unique according to Cauchy's theorem. We know the solution with initial data \((u_0, v_0, 0, \lambda)\). For any \( \lambda \), whatever \( t \), \( \frac{du}{dt} \) = 0. Thus if \( \frac{dv}{dt} \neq 0 \) at some point, then \( \frac{dv}{dt} \) does not vanish.

i) The geodesics whose equations are \( u = u_0, v = v_0 + \mu t \) are those which are in a plane through Oz. For the other geodesics, \( \frac{du}{dt} \) never vanishes and we can choose \( u \) as parameter. Let \( v' \) and \( v'' \) denote \( \frac{dv}{du} \) and \( \frac{d^2v}{du^2} \) respectively. We have

\[
\frac{dv}{dt} = v' \frac{du}{dt}, \quad \frac{d^2u}{dt^2} = 2 \frac{b \sin v}{a + b \cos v} v' \left( \frac{du}{dt} \right)^2,
\]

\[
\frac{d^2v}{dt^2} = v'' \left( \frac{du}{dt} \right)^2 + v' \frac{d^2u}{dt^2}.
\]

Hence we find that

\[
\frac{d^2v}{dt^2} = v'' \left( \frac{du}{dt} \right)^2 + v' \frac{d^2u}{dt^2}.
\]

(E)

\[
v'' = -\frac{a + b \cos v}{b} \sin v - 2 \frac{b \sin v}{a + b \cos v} (v')^2.
\]

\( v'' \) always has the same sign as \( -\sin v \).

j) \( \psi(0, 0) = P \) is the point of coordinates \((a+b, 0, 0)\) in \( \mathbb{R}^3 \). The geodesic for which \( \left( \frac{du}{dt} \right)_P = 0 \) has for equation \( v = 0, u = \lambda t \); it is the circle of radius \( a + b \) centered at \( O \) in the plane \( z = 0 \). If we substitute \(-v\) for \( v \) in (E), (E) remains satisfied. Then \((u/u, v/-v)\) is a symmetry with respect to the plane \( z = 0 \); it permutes the geodesic for which \( \left( \frac{dv}{du} \right)_P = a \) with the one for which \( \left( \frac{dv}{du} \right)_P = -a \).

k) When \( u \to 0 \) we have \( v_\alpha(u) \sim \alpha u \) and

\[
v''_\alpha(u) \sim -\left( \frac{a + b}{b} + \frac{2b}{a + b} \alpha^2 \right) v_\alpha.
\]

Thus \( v''_\alpha(u) \) decreases from \( \alpha \) until zero if \( \alpha \) is small enough. On \([0, u_\alpha[\), \( v_\alpha \) is increasing, \( v_\alpha(u_\alpha) > 0 \) and \( v_\alpha(v) < \alpha u \). Then \( v_\alpha, v'_\alpha, v''_\alpha \) are of the order of \( \alpha \). Define \( w = \lim_{\alpha \to 0}(v_\alpha/\alpha) \); \( w \) satisfies the equation \( w'' = -\frac{x + b}{b} w \) and the initial conditions \( w(0) = 0, w'(0) = 1 \).
Thus

\[ w(u) = \sqrt{\frac{b}{a+b}} \sin \left( \sqrt{\frac{a+b}{b}} u \right). \]

\[ w' \] vanishes for the first time at \( u_0 = \frac{\pi}{2} \sqrt{\frac{b}{a+b}} \). If \( \alpha \) is small enough, then \( u_\alpha \) is close to \( u_0 \) and smaller than \( \pi/2 \). \( v_\alpha(u_\alpha) \) is close to \( \sqrt{\frac{b}{a+b}} \).

1) Since we have \( v_\alpha(0) = 0 \) and \( v'_\alpha(u_\alpha) = 0 \), by symmetry that implies \( v_\alpha(2u_\alpha) = 0 \). There are three geodesics from \( P \) to \( Q \): this geodesic, the one with \( \left( \frac{dw}{du} \right)_P = -\alpha \) and the arc of circle in the plane \( z = 0 \). They have the same length \( l_\alpha \). We saw that, when \( \alpha \) tends to zero, \( l_\alpha \to l_0 = 2(a + b)u_0 = \pi \sqrt{b(a + b)} \). We saw also that if \( \alpha \) is small enough, then \( u_\alpha \) exists and \( l_\alpha < (a + b)\pi \).

**Solution to Problem 5.43.**

Questions (a) and (b) are solved in Problem 5.32.

c) We have \( \mathcal{E}(\nu_i, Y) = 0 \). Since \( \tilde{D}\mathcal{E} = 0 \),

\[ \ell_i(X, Y) = \mathcal{E}(\tilde{D}_X \nu_i, Y) = -\mathcal{E}(\nu_i, \tilde{D}_X Y). \]

Thus

\[ \tilde{D}_X Y = D_X Y - \sum_{i=1}^{k} \ell_i(X, Y) \nu_i. \]

As the Euclidean connection is without torsion, \( \mathcal{E}(\nu, \tilde{D}_X Y) = \mathcal{E}(\nu_i, \tilde{D}_Y X) = -\ell_i(Y, X) \) since \( \mathcal{E}(\nu_i, [\tilde{X}, \tilde{Y}]) = 0 \). Thus \( \ell_i(Y, X) \) is a symmetric bilinear form.

d) We have

\[ \tilde{D}_X \tilde{D}_Y Z = \tilde{D}_X [D_Y Z - \sum_{i=1}^{k} \ell_i(Y, Z) \nu_i] \]

\[ = D_X D_Y Z - \sum_{i=1}^{k} \ell_i(X, D_Y Z) \nu_i \]

\[ - \sum_{i=1}^{k} \ell_i(Y, Z) \tilde{D}_X \nu_i - \sum_{i=1}^{k} [\tilde{D}_X \ell_i(Y, Z)] \nu_i. \]

The result follows since \( \mathcal{E}(\nu_i, T) = 0 \) for any \( 1 \leq i \leq k \).

e) We have

\[ R(X, Y, Z, T) = -g[R(X, Y)Z, T] \]
with \( R(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \). Moreover,
\[
\mathcal{E}(\tilde{D}_X \tilde{D}_Y Z - \tilde{D}_Y \tilde{D}_X Z - \tilde{D}_{[X,Y]}Z, T) = -\tilde{R}(X, Y, Z, T) = 0.
\]
The result follows from (d).

f) If \( k = 1 \) and \( n = 2 \), then
\[
R(X, Y, X, Y) = \ell(X, X)\ell(Y, Y) - [\ell(X, Y)]^2.
\]
Choose two orthonormal vectors \( X, Y \) such that \( \ell \) is diagonal and
\( R(X, Y, X, Y) = \lambda_1 \lambda_2 \). Thus \( R = 2\lambda_1 \lambda_2 \). If \( k = 1 \) and \( n > 2 \), choose a
basis \( X_i \) of \( T_Q(M) \) such that \( \ell \) is diagonal. Then \( R(X_i, X_j, X_i, X_j) = \lambda_i \lambda_j \). All these values cannot be negative, since two \( \lambda_i \) have the same
sign or some \( \lambda_i \) are zero.

**Solution to Problem 5.44.**

a) We have \( \frac{1}{n}(\varphi^+_h - \varphi^+_h) = \varphi^+_h[\frac{1}{n}(\varphi^+_h - \varphi^+_h)] \). Passing to the limit leads to
\( d(\varphi^+_h)/dt = \varphi^+_h L_X h \).

_Necessity._ If \( h \) is invariant under \( G \), the left hand side above is
zero by hypothesis; thus \( L_X h = 0 \).

_Sufficiency._ \( L_X h = 0 \) implies \( \varphi^+_h = \varphi^+_0 h = h \).

b) We have \( |\psi^* g| = e^{n/2}|g| \); thus
\[
\psi^* h = |\psi^* g|^{-1/n} \psi^* g = e^f g / e^f |g|^{1/n} = h.
\]

_Necessity._ If \( \psi \) is a conformal transformation, \( h \) is invariant under
\( G \). So \( L_X h = 0 \).

_Sufficiency._ \( L_X h = 0 \) implies \( \varphi^+_h = h \). Consequently, \( \psi^* g = |\psi^* g|^{-1/n} g |g|^{-1/n} \) is of the form \( e^f g \).

c) We have \( \frac{\partial \psi}{\partial t} = Y \). Since \( Y^n \neq 0 \), \( \frac{\partial \psi^n}{\partial t} \neq 0 \). So we can express
t as a function of the coordinates of \( x \), in a neighbourhood of \( P \),
in a unique way according to the implicit function theorem. Then \( x_0 = \psi_{-t}(x) \), and \( x^1, x^2, \ldots, x^{n-1}, t \) form a coordinate system on this
neighbourhood. Indeed, consider the map
\[
\Gamma : (x^1, x^2, \ldots, x^{n-1}, t) \rightarrow (\psi^1_t(x_0), \psi^2_t(x_0), \ldots, \psi^n_t(x_0))
\]
de fined on a neighbourhood of \( 0 \in \mathbb{R}^n \). Then
\[
(D\Gamma)_0 = \begin{pmatrix}
1 & 0 & \ldots & 0 & Y^1 \\
0 & 1 & \ldots & 0 & Y^2 \\
\vdots & \vdots & \ddots & 1 & \vdots \\
0 & 0 & \ldots & 0 & Y^n
\end{pmatrix}
\]
is invertible. Thus \( \Gamma \) is a diffeomorphism of a neighbourhood \( \theta \) of
\( 0 \in \mathbb{R}^n \) onto a neighbourhood of \( P \).
d) In this coordinate system,
\[ L_Y h = |g|^{-1/n} \left[ L_Y g - \frac{1}{n} |g|^{-1} L_Y |g| g \right] \]
\[ = |g|^{-1/n} \left[ L_Y g - \frac{1}{n} (g^{ij} L_Y g_{ij}) g \right]. \]

Let us compute $g^{ij} L_Y g_{ij}$. Since $Y = \frac{\partial}{\partial t}$, we have $Y^i = 0$ for $i < n$ and $Y^n = 1$. Then
\[ \nabla_i Y^i = \partial_i Y^i + \Gamma^i_{ij} Y^j = \Gamma^i_{in} = \frac{1}{2} g^{ij} [\partial_i g_{nj} + \partial_n g_{ij} - \partial_j g_{ni}] \]
and
\[ g^{ij} L_Y g_{ij} = g^{ij} \partial_n g_{ij} = 2 \Gamma^i_{in} = 2 \nabla_i Y^i \text{ since } g^{ij} = g^{ji}. \]

The equality is proved in a coordinate system; since it is a tensor equality, it is valid in any coordinate system.

**Solution to Problem 5.45.**

a) Choose $X$ such that $\|X\| = 1$; in this case $t$ is the arc length. Suppose that $\exp_P tX$ is defined for $0 \leq t < \tau$, $\tau$ being the largest real number having this property ($\tau$ exists according to Theorem 5.11). The proof is by contradiction; we suppose $\tau$ finite. Let $\{t_i\}$ be an increasing sequence such that $t_i \to \tau$. Then $Q_i = \exp_P t_i X$ ($i \in \mathbb{N}$) is a Cauchy sequence. Indeed, $d(Q_i, Q_j) \leq |t_i - t_j|$, and $\{t_i\}$ is a Cauchy sequence. Since $M$ is complete, there exists $Q \in M$ such that $d(Q, Q_i) \to 0$ when $i \to \infty$. The geodesic has an end point. $Q = \exp \tau X$.

Now we use the theorem on the exponential mapping at $Q$. There is a neighbourhood $\Omega$ of $Q$ where $\exp^{-1}_Q$ is a diffeomorphism. Since $\exp_P tX \to Q$ when $t \to \tau$, there exists an $s$ such that $\exp_P tX \in \Omega$ for $s \leq t < \tau$. Now we know that there is a unique geodesic $\gamma(u \to \gamma(u))$ from $\exp_P sX$ to $Q$ included in $\Omega$ ($\gamma(0) = Q$ and $\gamma(\tau - s) = \exp_P sX$). Because of the uniqueness, $\gamma(\tau - t) = \exp_P tX$. Let $Y = \left( \frac{d\gamma}{du} \right)_Q$; then $\gamma(u) = C(u, Q, Y)$ exists for any $u \in [-\epsilon, \epsilon], \epsilon > 0$ small. Thus the geodesic $C(s)$ extends for $\tau \leq s < \tau + \epsilon$. This is in contradiction with the definition of $\tau$.

b) Let $\gamma(u)$ be the geodesic of $W$ from $C(t)$ such that
\[ \left( \frac{d\gamma}{du} \right)_{C(t)} = \left( \frac{dC(t)}{dt} \right)_{C(t)}. \]

Since $W$ is compact, $W$ is complete and $\gamma(u)$ extends itself infinitely. According to the hypothesis $\gamma(u)$ is a geodesic of $M$, but there is only one geodesic with given initial data $C(t)$, namely $(dC(t)/dt)_{C(t)}$. Thus $C$ and $\gamma$ are the same.
c) We have $dC/dt = \partial/\partial x^1$, so

$$D_{\frac{\partial}{\partial x^1}} \left( \frac{\partial}{\partial x^i} \right) = \Gamma^j_{1i} \frac{\partial}{\partial x^j} = 0,$$

since the Christoffel symbols vanish on $C$. The vector fields $\partial/\partial x^i$ are parallel along $C$.

d) The map $W \times V \ni (P, Q) \rightarrow d(P, Q) > 0$ is continuous on a compact set. Thus $\tau$ is achieved: There exist $P_o$ and $Q_o$ such that $d(P_o, Q_o) = \tau > 0$.

e) Let $X(t)$ be a parallel vector field along $C$. The norm of $X(t)$ is constant along $C$. Indeed, $g_{C(t)}(X(t), X(t)) = \text{Constant}$ since the covariant derivatives $D_{dC/dt}$ of $g$ and $X(t)$ are zero. So the map $T_{P_o}(M) \ni X(0) \rightarrow X(\tau) \in T_{Q_o}(M)$ is injective; it is an isomorphism. Thus $\dim H = p$. Since $\dim T_{Q_o}(V) = q$ with $p + q \geq n$, it follows that $H \cap T_{Q_o}(V)$ does not reduce to zero, because $p + q - (n - 1) > 0$. Indeed, $(\partial_{\partial x}^i)_{Q_o}$ is perpendicular to $H$ and to $T_{Q_o}(V)$ according to Proposition 5.13. Thus $\dim (H + T_{Q_o}(V)) \leq n - 1$.

f) $t \rightarrow X(t)$ and $t \rightarrow C(t)$ are $C^\infty$ differentiable, and we also know that $(R, Z) \rightarrow \exp_{R} Z$ is $C^\infty$ differentiable. Thus $(t, \lambda) \rightarrow \gamma_{\lambda}(t)$ is $C^\infty$ differentiable. Also, $\lambda \rightarrow \exp_{Q_o} \lambda Y$ is a geodesic of $M$, included in $V$ since $Y \in T_{Q_o}(V)$. Likewise $\lambda \rightarrow \exp_{P_o} \lambda X(0)$ is a geodesic of $M$, included in $W$ since $X(0) \in T_{P_o}(W)$. Thus $f(\lambda) \geq \tau$.

g) The function

$$\lambda \rightarrow f(\lambda) = \int_0^T \sqrt{g_{ij}(\gamma_{\lambda}(t))} \frac{dy_{\lambda}^j}{dt} \frac{dy_{\lambda}^i}{dt} \, dt$$

is $C^\infty$ differentiable. Since $f(\lambda) \geq f(0)$, we have $f'(0) = 0$. Choose $\lambda$ small enough so that the geodesic $\gamma_{\lambda}$ lies in the coordinate system. Then $\gamma_0(t) = C(t)$; thus $\gamma_{\lambda}^i(t) = t$ and $\gamma_{\lambda}^i(t) = 0$ for $i > 1$. However,

$$\left( \frac{\partial \gamma_{\lambda}(t)}{\partial \lambda} \right)_{\lambda=0} = X(t) \quad \text{and} \quad X(t) = \sum_{i=2}^n X^i \left( \frac{\partial}{\partial x^i} \right)_{C(t)}$$

with each $X^i = \text{Constant}$, since $g(X(t), \frac{dC/dt}{dt}) = 0$ and according to c). We have

$$\frac{\partial}{\partial \lambda} \left[ g_{ij}(\gamma_{\lambda}(t)) \frac{dy_{\lambda}^j}{dt} \frac{dy_{\lambda}^i}{dt} \right]_{\lambda=0} = 0 \quad \text{and} \quad g_{ij}(\gamma_0(t)) \frac{dy_0^j}{dt} \frac{dy_0^i}{dt} = 1$$

since $(\partial_k g_{ij})_{C(t)} = 0$ (the Christoffel symbols vanish on $C$). Since $\lambda \rightarrow \gamma_{\lambda}(t)$ satisfies the geodesic equation, we have

$$\left( \frac{\partial^2 \gamma_{\lambda}^i(t)}{\partial \lambda^2} \right)_{\lambda=0} = -\Gamma^i_{jk}(\gamma_0(t)) \left( \frac{dy_{\lambda}^j}{dt} \right)_{\lambda=0} \left( \frac{dy_{\lambda}^k}{dt} \right)_{\lambda=0} = 0,$$
since the Christoffel symbols vanish on $C$. So when we compute $f''(0)$, we can get something which is not zero only if we differentiate $g_{ij}$ twice. But as $d\gamma^k_0/dt = 0$ when $i > 1$, we obtain

$$f''(0) = \frac{1}{2} \int_0^T \partial_{ki} g_{11}(C(t)) X^k(t) X^i(t) dt.$$ 

Since $\partial_t g_{jk}(C(t)) = 0$, it follows that $(\partial_{i1} g_{jk}) C(t) = 0$. So if $l = j = 1$ in the expression of $R_{kiij}$, we get

$$\sigma \left( X(t), \frac{dC}{dt} \right) = R_{k111}(C(t)) X^k(t) X^i(t) = -\frac{1}{2} \partial_{ik} g_{11} X^i(t) X^k(t)$$

since $X(t)$ and $\frac{dC}{dt}$ are unit vectors $(g_{ik}(C(t)) X^i(t) X^k(t) = 1)$.

h) If $\sigma(X(t), \frac{dC}{dt}) > 0$, then

$$f''(0) = - \int_0^T \sigma \left( X(t), \frac{dC}{dt} \right) dt < 0,$$

which is a contradiction. Indeed, $f$ is minimum at 0; thus $f''(0) \geq 0$. So in that case $W_p \cap V_q \neq \emptyset$.

Solution to Problem 5.46.

a) Let $[a, b] \ni t \rightarrow \tilde{C}(t)$. Set $C(t) = \pi \circ \tilde{C}(t)$. Locally

$$\tilde{g} \left( \frac{d\tilde{C}}{dt}, \frac{d\tilde{C}}{dt} \right) = g \left( \frac{dC}{dt}, \frac{dC}{dt} \right),$$

since any point $\tilde{P} \in W$ admits a neighbourhood diffeomorphic to a neighbourhood of $P = \pi(\tilde{P})$. Thus $\tilde{L}(C) \geq L(C)$. We can have strict inequality, since some arcs of $\tilde{C}$ may have the same projection. If $\tilde{C}$ satisfies the geodesic equation, so does $C$ (it is a local condition).

b) Let $\{\tilde{P}_i\}$ be a Cauchy sequence in $(W, \tilde{g})$. According to (a), $d(\tilde{P}_i, \tilde{P}_j) \geq d(P_i, P_j)$ with $P_i = \pi(\tilde{P}_i)$. Thus $\{P_i\}$ is a Cauchy sequence in $M$, which is compact. Let $P$ be the limit of $\{P_i\}$. $P$ admits a neighbourhood $B(\rho)$ of radius $\rho$ centered at $P$ such that $\pi^{-1}(B(\rho))$ is diffeomorphic to $B(\rho) \times F$, with $F$ a discrete space, by definition of a covering manifold. There exists $k$ such that if $i \geq k$ and $j \geq k$ then $d(\tilde{P}_i, \tilde{P}_j) < \rho/2$. $\tilde{P}_i$ belongs to an open set $\tilde{\Omega}$ of $B(\rho) \times F$ which is diffeomorphic to $B(\rho)$ by $\pi/\tilde{g}$. Since $d(\tilde{P}_i, \tilde{P}_j) < \rho/2$, all points $\tilde{P}_j$ for $j \geq k$ belong to $\tilde{\Omega}$. Thus there exists $P' \in \tilde{\Omega} \cap \pi^{-1}(P)$ such that $\tilde{P}_j \rightarrow P'$ when $j \rightarrow \infty$.

c) Any point of $C$ admits a neighbourhood $\Omega$ such that $\pi^{-1}(\Omega)$ is diffeomorphic to $\Omega \times F$. Since $C$ is compact, a finite number of these neighbourhoods, $\Omega_1, \Omega_2, \ldots, \Omega_k$, cover $C$. $\tilde{P}$ belongs to $\tilde{\Omega}_1$, and is
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diffeomorphic to \( \Omega_1 \) by \( \pi/\tilde{\Omega}_1 \). Let \( \Phi_1 \) be the inverse of \( \pi/\tilde{\Omega}_1 \). Then \( \Phi_1(\Omega \cap \Omega_1) \) is an arc \([a, a_1] \to \tilde{C} \to W \). We choose

\[
\tilde{C}(a_1) \in \pi^{-1}(\Omega_2) \cap \Phi_1(\Omega \cap \Omega_1).
\]

\( \tilde{C}(a_1) \) belongs to \( \tilde{\Omega}_2 \) and is diffeomorphic to \( \Omega_2 \) by \( \pi/\tilde{\Omega}_2 \). And so on. We construct an arc \( \tilde{C} \) of \( M \) which is locally diffeomorphic to \( C \). If \( \tilde{C}(b) = \tilde{P} \), \( \tilde{C} \) would be closed, thus homotopic to zero since \( W \) is simply connected. By \( \pi \), we would have \( C \) homotopic to zero, which is in contradiction with the hypothesis. Indeed, let \( \tilde{F}(t, \lambda) \) be a continuous map of \([a, b] \times [0, 1]\) into \( W \) such that \( \tilde{F}(t, 0) = \tilde{C}(t) \) and \( \tilde{F}(t, 1) = \tilde{P} \), for instance. Then \( (t, \lambda) \to F(t, \lambda) = \pi \circ \tilde{F}(t, \lambda) \) would be a continuous map of \([a, b] \times [0, 1]\) into \( M \) such that \( F(t, 0) = C(t) \) and \( F(t, 1) = P \).

d) Since \( C_0 \) is homotopic to \( C \), there exists a continuous map \( G(t, l) \) of \([a, b] \times [0, 1]\) into \( M \) such that \( G(t, 0) = C(t) \) and \( G(t, 1) = C_0(t) \). \( \Omega_1, \ldots, \Omega_k \) is a covering of \( C \). For \( \lambda \) small enough (\( \lambda \leq \epsilon_1 \)), \( \Omega_1, \ldots, \Omega_k \) is also a covering of \( G([a, b] \times [0, \epsilon_1]) \). Set \( \tilde{G}(t, l) = \Phi_1 \circ G(t, l) \) for \((t, l) \in [0, a_1] \times [0, \epsilon_1] \), \( \tilde{G}(t, l) = \Phi_2 \circ G(t, l) \) for \((t, l) \in [a_1, a_2] \times [0, \epsilon_1] \), and so on. Then we consider the curve \( C_{\epsilon_1} = G(t, \epsilon_1) \). From \( C_{\epsilon_1} \), we obtain \( C_{\epsilon_2} \) just as we obtained \( C_{\epsilon_1} \) from \( C \). And so on. Thus there exists a continuous map \( \tilde{G}(t, l) \) of \([a, b] \times [0, 1]\) into \( W \) such that \( \tilde{G}(t, 0) = \tilde{C}(t) \) and \( \tilde{G}(t, 1) = \tilde{C}_0(t) \). Then \( l \to \tilde{G}(b, l) \) is a continuous function in \( \pi^{-1}(P) = F \); thus this function is constant, \( \tilde{C}_0(b) = \tilde{C}(b) = \tilde{P} \).

e) If we choose \( R \) close to \( P \) and \( \tilde{R} \) close to \( \tilde{P} \), by the construction above \( \tilde{\gamma} \) will be close to \( \tilde{C} \). Thus \( \tilde{R}' \) is close to \( \tilde{P}' \) and \( R \to f(R) \) is continuous. Since \( f(R) > 0 \) and \( M \) is compact, \( f(R) \) achieves its minimum at least at one point \( Q \in M \).

f) We saw (question a)) that \( \pi(\tilde{\gamma}) \) is a geodesic from \( Q \) to \( Q \), and its length is \( f(Q) \). Moreover \( \pi(\tilde{\gamma}) \) is also a geodesic at \( Q \), as otherwise there exists a curve \( \gamma' \) close to \( \pi(\tilde{\gamma}) \), thus homotopic to \( \pi(\tilde{\gamma}) \), such that its length is smaller than \( f(Q) \). This is impossible, because then the curve \( \gamma' \) constructed above would be strictly smaller than \( f(Q) \).

We have proved the existence of a shortest closed geodesic in any homotopy class.
Chapter 6

The Yamabe Problem:
An Introduction to Research

6.1. For less than forty years, analysis has been used in order to try to solve problems in geometry. A pioneering work has been completed by Yamabe [16], on the scalar curvature problem. In Chapter 5, we have acquired enough knowledge to study nonlinear problems in Riemannian geometry. Nevertheless, for solving them it is necessary to know some theorems in functional analysis, at least three of which we will give below. For example, let us study the Yamabe problem.

6.2. The Yamabe problem. Yamabe wanted to prove that on a compact Riemannian manifold \((M, g)\) of dimension \(n \geq 3\) there always exists a metric with constant scalar curvature. This is a geometrical problem which we will transform into a problem in analysis. Let \(R\) be the scalar curvature of \((M, g)\). We suppose \(R\) is not constant—otherwise we have nothing to do. If we think of all possible changes of metric, the simplest is a conformal one. Therefore Yamabe considered the conformal metric \(g' = e^f g\), where \(f\) is a \(C^\infty\) function on \(M\).

6.3. Let us compute \(R'\) in terms of \(f\). If we are in a local chart, \(\Gamma^l_{ik}\) and \(\Gamma^l_{ik}\) denote the Christoffel symbols corresponding to \(g'\) and \(g\) respectively:

\[
\Gamma^l_{ik} - \Gamma^l_{ik} = \frac{1}{2} g^{lm}(g_{mk}\partial_l f + g_{mi}\partial_k f - g_{ik}\partial_m f),
\]

according to 5.5.
To simplify we assume that the coordinates are normal at $P$ for $g$ ($g_{ij}(P) = \delta^j_i$ and $\Gamma^k_{ij}(P) = 0$) (see 5.7):

$$R^l_{kij} - R^l_{kij} = \partial_i(\Gamma^{il}_{jk} - \Gamma^{il}_{jk}) - \partial_j(\Gamma^{il}_{ik} - \Gamma^{il}_{ik}) + \Gamma^{il}_{im}\Gamma^{lm}_{jk} - \Gamma^{il}_{jm}\Gamma^{lm}_{ik}.$$ 

Thus at $P$

$$R'_{kij} - R_{kij} = \partial_i(\Gamma'_{jk} - \Gamma'_{jk}) - \partial_j(\Gamma'_{ik} - \Gamma'_{ik}) + \Gamma'_{im}\Gamma'_{jk} - \Gamma'_{jm}\Gamma'_{ik},$$

$$R'_{kij} - R_{kij} = \frac{1}{2}g^{jm}(g_{mk}\partial_{ij}f + g_{mj}\partial_{ik}f - g_{jk}\partial_{im}f)$$

$$- \frac{1}{2}g^{jm}(g_{mk}\partial_{ij}f + g_{mi}\partial_{jk}f - g_{ik}\partial_{jm}f)$$

$$+ \frac{1}{4}(\partial_m f + n\partial_m f - \partial_m f)g^{ml}(g_{lk}\partial_j f + g_{lj}\partial_k f - g_{jk}\partial_l f)$$

$$- \frac{1}{4}(g_{im}\partial_j f + g_{lj}\partial_m f - g_{jm}\partial_l f)(g_{ik}\partial_k f - g_{iq}\partial_k f - g_{ik}\partial_q f)g^{il}g^{mk},$$

$$R'_{kij} - R_{kij} = \frac{1}{2}(2\partial_k f + \Delta f g_{jk}) - \frac{1}{2}(n\partial_{kj} f) + \frac{n}{4}(2\partial_k f \partial_j f - \nabla^\nu f \nabla_\nu f g_{jk})$$

$$- \frac{1}{4}(g^{iq}\partial_j f + \delta_j^i \nabla^q f \nabla^i f \delta_j^q)(g_{qk}\partial_i f + g_{qi}\partial_k f - g_{ik}\partial_q f),$$

$$R'_{kij} - R_{kij} = -\frac{n-2}{2}\partial_{kj} f + \frac{1}{2}\Delta f g_{jk} + \frac{n}{2}\partial_k f \partial_j f - \frac{n}{4}\nabla^\nu f \nabla_\nu f g_{jk}$$

$$- \frac{1}{4}(n\partial_j f \partial_k f + 2\partial_k f \partial_j f - 2\nabla^\nu f \nabla_\nu f g_{jk}),$$

and

$$R'_{kij} - R_{kij} = -\frac{n-2}{2}\nabla_k \nabla_j f + \frac{1}{2}\Delta f g_{jk}$$

$$+ \frac{n-2}{4}\nabla_k f \nabla_j f - \frac{n-2}{4}\nabla^\nu f \nabla_\nu f g_{jk}. $$

We have equality between two tensor fields, so the identity holds in any coordinate system. Thus, contracting (1) by $g^{kj}$, we obtain, since $g'^{ij} = e^{-f}g^{ij}$,

$$R' e^f - R = (n-1)\Delta f - \frac{(n-2)(n-1)}{4}\nabla^\nu f \nabla_\nu f.$$ 

On this expression we do not see how to proceed to find a function $f$ such that $R'$ is constant. To simplify the equation, Yamabe considered the conformal deformation in the form $g' = \varphi^{4/(n-2)}g$. Now $\varphi$ must be strictly positive.
Letting \( f = \frac{4}{n-2} \log \varphi \) in (2), we find that

\[
\frac{4(n-1)}{(n-2)} \Delta \varphi + R \varphi = R' \varphi^{\frac{n+2}{n-2}} \quad \text{with} \quad \varphi > 0.
\]

This is an elliptic equation (the Laplacian operator is elliptic) which is nonlinear: the exponent in the right hand side is greater than 1.

6.4. The Yamabe problem is a geometrical one: find a metric with constant scalar curvature. We have proven that if we look for a conformal metric, the problem is equivalent to proving that equation (3) with \( R' = \text{Constant} \) has a strictly positive \( C^\infty \) solution. It is a problem of PDEs (PDEs means partial differential equations).

It is easy to see that, if there are two solutions of (3), the new scalar curvatures of them have the same sign. Indeed, assume that \( g' \) has constant scalar curvature \( R' \) and \( g = \gamma^{4/(n-2)}g \) has constant scalar curvature \( \tilde{R} \). Let us compute \( R' \) in the metric \( \tilde{g} \). If we set \( \varphi = \gamma \psi \), this is possible since \( \varphi \) and \( \gamma \) are strictly positive. Thus \( g' = \psi^{4/(n-2)}\tilde{g} \) and

\[
4 \frac{\tilde{R}}{(n-2)} \tilde{\Delta} \psi + \tilde{R} \psi = R' \psi^{(n+2)/(n-2)}.
\]

Here \( \tilde{\Delta} \psi = -\tilde{g}^{ij} \tilde{\nabla}_i \tilde{\nabla}_j \psi = -\tilde{g}^{ij} (\partial_{ij} \psi - \tilde{\Gamma}^k_{ij} \partial_k \psi) \). Now let us integrate (4) with respect to the metric \( \tilde{g} \):

\[
\tilde{R} \int \psi d\tilde{V} = R' \int \psi^{(n+2)/(n-2)} d\tilde{V}
\]

since \( \int \tilde{\Delta} \psi d\tilde{V} = 0 \).

Thus \( \tilde{R} \) and \( R' \) have the same sign (or are both equal to zero).

If \( \tilde{R} = R' = 0 \), then \( \tilde{\Delta} \psi = 0 \); hence \( \psi = \text{Constant} \). The solutions of (4) are proportional. If \( \tilde{R} = R' < 0 \) we easily see that (4) has only one solution \( \psi \equiv 1 \). Indeed, at a point \( P \) where \( \psi \) is maximum we have \( \tilde{\Delta} \psi > 0 \); thus \( R' \psi(P)^{\frac{n+2}{n-2}} \geq \tilde{R} \psi(P) \), and we find that \( \psi(P)^{4/(n-2)} \leq 1 \). Now at a point \( Q \) where \( \psi \) is minimum we have \( \tilde{\Delta} \psi \leq 0 \); thus \( R' \psi(Q)^{\frac{n+2}{n-2}} \leq \tilde{R} \psi(Q) \), and we find that \( \psi(Q)^{4/(n-2)} \geq 1 \). Consequently \( \psi(P) = \psi(Q) = 1 \).

We have proven that there are three cases, according to whether \( R' \) is positive, negative or zero, and in the negative and zero cases the solutions of (4) are proportional. In the negative case, if \( \tilde{R} \) and \( R' \) are constant, then \( \psi = (\tilde{R}/R')^{(n-2)/4} \) is the unique solution of (4).

6.5. The variational method. How can we prove the existence of a solution of (3) with \( R' = \text{Constant} \)? There are several methods for solving nonlinear equations; one of the most powerful is the variational method. We have to consider, on a set \( \mathcal{A} \) of functions, a functional \( I \) bounded from below on \( \mathcal{A} \) such that the Euler equation of the variational problem
is (3). Then, \( \{u_i\} \ (i \in \mathbb{N}) \) being a minimizing sequence (i.e. \( u_i \subset \mathcal{A} \) and \( \lim_{i \to \infty} I(u_i) = \mu \), the inf of \( I(u) \) on \( \mathcal{A} \)), we try to exhibit a subsequence \( \{u_j\} \subset \{u_i\} \) which converges to a strictly positive solution of (3) with \( R' = \text{Constant} \).

This is roughly the idea of the method. In practice, it is not so easy, as we will see.

6.6. For some equations, we can imagine several functionals; for others there are none. Here \( R' \) appears as a Lagrange multiplier.

The constraint which will give \( \psi^{(n+2)/(n-2)} \) is

\[
K(u) = \frac{1}{N} \int_M |u|^N dV \quad \text{with} \quad N = \frac{2n}{n-2}.
\]

Indeed, \( D_u[(u^2)^{N/2}](v) = \frac{N}{2} (u^2)^{\frac{N}{2}-1} 2uv = N|u|^{4/(n-2)}uv \).

The functional which will give twice the left hand side of (3) is

\[
I(u) = \int_M \left( \frac{4(n-1)}{(n-2)} \nabla_i u \nabla_i u + Ru^2 \right) dV.
\]

Indeed,

\[
D_u I(v) = 2 \int_V \left( \frac{4(n-1)}{(n-2)} \nabla_i u \nabla_i v + Ru \right) dV.
\]

If we formally perform an integration by parts, we obtain

\[
D_u I(v) = 2 \int_M \left( \frac{4(n-1)}{(n-2)} \Delta u + Ru \right) v dV.
\]

We have to use the fact that \( D_u I(v) \) is proportional to \( D_u K(u) \). Thus the Euler equation is

\[
(6) \quad \frac{4(n-1)}{(n-2)} \Delta u + Ru = \mu |u|^{4/(n-2)} u
\]

if the constraint is \( K(u) = 1/N \). Indeed, \( \mu \) is the Lagrange multiplier: multiplying (6) by \( u \) and integrating lead to \( I(u) = \mu N K(u) \).

6.7. The Sobolev imbedding theorem. Now we have to choose the set \( \mathcal{A} \). Assume that we choose for \( \mathcal{A} \) the set of \( C^\infty \) strictly positive functions \( \varphi \) such that

\[
\|\varphi\|_N = \left( \int |\varphi|^N dV \right)^{1/N} = 1.
\]

It will be very difficult, even impossible, to prove that \( u \), the limit of a subsequence \( \{u_j\} \), is a strictly positive \( C^\infty \) function. On the other hand, we can write the Euler equation only if \( u \in \mathcal{A} \). For this reason, we introduce the Sobolev space \( H^2_1 \).
Let \( E \) be the set of \( C^\infty \) functions on \( M \) endowed with the norm
\[
\|u\|_{H^1} = (\|\nabla u\|^2 + \|u\|^2)^{1/2}.
\]
The gradient is \( |\nabla u| = (\nabla_i u \nabla^i u)^{1/2} \); its \( L^2 \) norm is \((\int_M |\nabla u|^2 dV)^{1/2}\).

The Sobolev space \( H^1 \) is the completion of \( E \) with respect to the norm \( \| \cdot \|_{H^1} \). It is a Hilbert space. Moreover, the Sobolev imbedding theorem (see [2]) asserts that \( H^1 \subset L^N \) for compact Riemannian manifolds, and that the inclusion is continuous (i.e. there exists a constant \( C \) such that any \( u \in H^1 \) satisfies \( \|u\|_N \leq C\|u\|_{H^1} \)). To solve the variational problem, we will choose \( \mathcal{A} = \{u \in H^1, u \geq 0 \mid \|u\|_N = 1 \} \), which makes sense according to the Sobolev imbedding theorem.

In general, choosing a constraint like \( u \geq 0 \) implies several difficulties (we cannot write the Euler equation when \( u \) is zero). These difficulties are not present here. Indeed, it is a fact that if \( u \in H^1 \), then \( |u| \in H^1 \) and \( |\nabla|u| = |\nabla u| \) almost everywhere; we have \( I(u) = I(|u|) \), and obviously \( K(u) = K(|u|) \).

Thus the inf of \( I(u) \) on \( \mathcal{A} \) is equal to the inf of \( I(u) \) on \( \tilde{\mathcal{A}} = \{u \in H^1, u \geq 0 \mid \|u\|_N = 1 \} \). Therefore, the Euler equation can be written without any technical problem, and the limit \( u \) will be positive or zero: \( u = |u| \). Hence equation (6) is equation (3) with \( R' = \mu \).

6.8. We therefore consider the following variational problem: find \( \inf I(u) \) for \( u \in \mathcal{A} \). Recall that
\[
I(u) = 4\frac{(n-1)}{(n-2)} \int_M |\nabla u|^2 dV + \int_M R u^2 dV
\]
and \( \mathcal{A} = \{u \in H^1, u \geq 0 \mid \|u\|_N = 1 \} \).

Let us prove that \( \mu = \inf_{\mathcal{A}} I(u) \) is finite. Observe that
\[
I(u) \geq \inf(R, 0)\|u\|^2 \geq \inf(R, 0)V^{2/n}\|u\|^2_V = \inf(R, 0)V^{2/n},
\]
according to Hölder's inequality. Here \( V = \int_M dV \); without loss of generality we may suppose that the volume \( V \) is equal to 1. Indeed, by a homothetic change of metric we can set the volume equal to 1, and a homothetic change of metric is a conformal one. Let \( g' = kg \) with \( k > 0 \); then
\[
V' = \int dV' = k^{n/2} \int dV = k^{n/2}V.
\]
We have only to choose \( k = V^{-2/n} \). Henceforth we assume that the volume is equal to 1. Let us consider a minimizing sequence \( \{u_i\} \subset \mathcal{A} \): \( I(u_i) \to \mu \). We can suppose \( I(u_i) \leq \mu + 1 \). According to Hölder's inequality, \( \|u_i\|_2 \leq \|u_i\|_N = 1 \). Moreover,
\[
4\frac{(n-1)}{(n-2)} \|\nabla u_i\|^2 \leq \mu + 1 - \inf(R, 0)\|u_i\|^2_V \leq \text{Constant}.
\]
Thus the sequence \( \{u_i\} \) is bounded in \( H^2_1 \).

### 6.9. Weak convergence

The Banach theorem (see [2]) asserts, in particular, that in a Hilbert space the closed unit ball is weakly sequentially compact. This implies, for every bounded sequence \( \{u_i\} \) in \( H^2_1 \), that there exist \( w \in H^2_1 \) and a subsequence \( \{u_{j}\} \) of \( \{u_i\} \) such that \( u_j \rightharpoonup w \) weakly in \( H^2_1 \) (i.e., for any \( u \in H^2_1 \), \( \langle u, u_j \rangle_{H^2_1} \to \langle u, w \rangle_{H^2_1} \) as \( j \to \infty \)). It is easy to verify that convergence in \( H^2_1 \) (\( \|u_j - w\|_{H^2_1} \to 0 \)) implies weak convergence.

We say that \( w \) is the weak limit in \( H^2_1 \) of the sequence \( \{u_j\} \). One can prove that \( \|\text{weak lim }u\| \leq \liminf_{j \to \infty} \|u_j\|_{H^2_1} \); thus

\[
(7) \quad \|w\|_{H^2_1} \leq \liminf_{j \to \infty} \|u_j\|_{H^2_1}.
\]

### 6.10. The Yamabe Problem

We have used two theorems, the Sobolev theorem and the Banach theorem; we now have to use a third theorem, the Kondrakov theorem (see [2]). It asserts that, on compact Riemannian manifolds, the inclusion \( H^2_1 \subset L_q \) is compact for \( 1 \leq q < N = 2n/(n - 2) \). This means that, if \( \{u_i\} \) is a bounded sequence in \( H^2_1 \), there exists a subsequence which converges in \( L_q \).

Returning to our problem, according to the Kondrakov theorem, we can choose the subsequence \( \{u_j\} \) such that \( u_j \rightharpoonup \bar{w} \) in \( L_2 \) and also \( u_j \to \bar{w} \) a.e. It is not difficult to prove that in fact \( w = \bar{w} \). Thus we have \( \|u_j\|_2 \to \|w\|_2 \), \( \|\nabla w\|_2 \leq \liminf_{j \to \infty} \|\nabla u_j\|_2 \) according to (7), and

\[
\int_V Ru_j^2 dV \to \int_V Rw^2 dV.
\]

Indeed,

\[
\left| \int_V Rw^2 dV - \int_V Ru_j^2 dV \right| \leq \sup |R| \|w - u_j\|_2 (\|w\|_2 + \|u_j\|_2) \to 0.
\]

We infer that \( I(w) \leq \mu = \lim I(u_j) \). Assume that we are able to prove that \( \|w\|_N = 1 \). Then \( w \in A \) and \( I(w) = \mu \) according to the definition of \( \mu \) (if \( w \in A \), \( I(w) \geq \mu \)). We then write the Euler equation.

### 6.11. Regularity

So we have exhibited \( w \in H^2_1, w \geq 0, w \not= 0 \), which satisfies equation (6) weakly in \( H^2_1 \). That is, for any \( v \in H^2_1 \),

\[
4\left(\frac{n-1}{n-2}\right) \int_M \nabla^i w \nabla_i v dV + \int_M RwvdV = \mu \int_M w^{N-1}vdV.
\]

The maximum principle then implies \( w > 0 \), and some regularity theorems imply \( w \in C^\infty \). Thus \( w \) satisfies equation (6) in the sense of functions.

Unfortunately it is impossible to prove that \( \|w\|_N = 1 \). This is why Yamabe considered an approximated equation

\[
(8) \quad 4\left(\frac{n-1}{n-2}\right) \Delta u + Ru = \mu q u^{q-1}, \quad u > 0 \text{ for } 2 < q < N.
\]
For this equation the proof above works. Indeed, now
\[ A_q = \{ u \in H^2, u \geq 0 \mid \|u\|_q = 1 \}. \]

According to the Kondrakov theorem, we can choose the subsequence \( \{u_j\} \) such that \( u_j \to w_q \) in \( L_q \). Consequently 1 = \( \|u_j\|_q \to \|w_q\|_q \). Thus \( w_q \in A \), and \( w_q \) satisfies equation (8). Now we have to see what happens when \( q \to N \). This is difficult. Nevertheless, we have succeeded in proving the following.

6.12. Theorem (Aubin; see [1] or [2]). We always have \( \mu \leq n(n-1)\omega_n^{2/n} \).
If \( \mu < n(n-1)\omega_n^{2/n} \), there exists a strictly positive solution \( w \in C^\infty \) of (3) with \( R' = \mu \) and \( \|w\|_N = 1 \). Here \( \omega_n \) is the volume of the sphere of radius 1 and dimension \( n \). \( \mu \) is defined in 6.8.

In order to see that the inequality of Theorem 6.12 is strict, we have to put test functions in the functional \( I \). If the manifold is not conformal to the sphere, we can prove that \( \mu < n(n-1)\omega_n^{2/n} \). Thus the Yamabe problem is solved. There always exists a conformal metric \( g' \) such that \( R' = \text{Constant} \) (the sphere has constant curvature, hence constant scalar curvature).

6.13. Perspective in research. On compact Riemannian manifolds, the Yamabe problem is solved. One may pose the same problem on complete noncompact manifolds. There are only a few results, and they are different from those on compact manifolds. For instance, it is not clear that there are three different cases, positive, negative or zero (according to the sign of \( \mu \) in the compact case). We can also consider generalised versions of the Yamabe problem. For example, let us consider the prescribed scalar curvature problem:

Let \( f \) be a \( C^\infty \) function on the Riemannian manifold \((M_n, g)\). We ask the following question: Does there exist a conformal metric \( g' \) for \( g \) such that \( R' = f \)?

In dimension \( n \geq 3 \), this problem is equivalent to solving the following equation:

\[
4 \frac{(n-1)}{(n-2)} \Delta \varphi + R \varphi = f \varphi^{(n+2)/(n-2)}, \quad \varphi > 0.
\]

In dimension \( n = 2 \), the equation to solve is (see (2) with \( n = 2 \))

\[
\Delta \varphi + R = f e^{\varphi}.
\]

This problem is particularly hard on the sphere, where it is the so-called Nirenberg problem.
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Notation

Basic Notation

We use the Einstein summation convention.
Positive means strictly positive.
Nonnegative means positive or zero.
Compact manifold means compact manifold without boundary unless we say otherwise.
N is the set of positive integers, n ∈ N.
R^n is the Euclidean n-space, n ≥ 2, with points x = (x1, x2, ..., xn), x_i ∈ R, the set of real numbers.
When it is not otherwise stated, a coordinate system (x1, x2, ..., xn) in R^n (or (x, y, z, t) in R^4) is chosen to be orthonormal.
We often write ∂_i for ∂/∂x_i and ∂_i∂_j for ∂_i∂_j.
Sometimes we write ∇_ij for ∇_i∇_j.
[a, b] or (a, b) means an open interval in R. (a, b) may also be the point of R^2 whose coordinates are a and b.

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C^k, C^∞, C^w differentiable function. 0.23, 0.26
C^w_p = n!/(n - p)!pl manifold. 1.6
d exterior differential. 2.24, 2.25
df differential of f. 0.26
dV Riemannian volume element. 5.23
D_X Y or D(X, Y) 4.2
dx^j 0.21, 2.25
d(P, Q) distance from P to Q. 5.4
E = {x ∈ R^n | x^i < 0} 2.34
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f'(x) differential of f at x. 0.23
g Riemannian metric. 5.1
g_1, g_2 components of g 5.5
L_X Lie derivative with respect to X. 3.4
M_n or M manifold of dimension n. 1.1
(M_n, g) Riemannian manifold. 5.1
O(n) 2.45
P_n(R) real projective space. 1.9
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R_{ijkl} 5.8
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S_n the sphere of dimension n
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\begin{align*}
T_P(M) &= 2.3 \\
T(M) &= 2.5 \\
T^*(M) &= 2.5 \\
T(X,Y) &= 4.6 \\
T(n,p) &= 1.20 \\
T(n,p,k) &= 1.21 \\
T^*_p(M) &= 2.14 \\
\Gamma(M) &= \text{space of vector fields on } M. \ 2.14 \\
\Gamma^*_{ij} &= \text{Christoffel symbols. } 4.3, 4.5, 5.5 \\
\Delta &= \text{Laplacian operator. } 5.18 \\
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\Phi^* &= 2.7, 2.23 \\
\chi &= \text{the Euler-Poincaré characteristic. } 5.22 \\
\omega_n &= \text{the volume of the unit sphere } S_n \\
\nabla &= 4.13 \\
\nabla Y &= \nabla_i Y^j dx^i \otimes \partial_j \ 4.4 \\
(\partial_i \partial^j)_P &= \text{tangent vector at } P. \ 2.3 \\
[X,Y] &= \text{bracket. } 2.15 \\
* &= \text{adjoint operator. } 5.1 \\
\otimes &= \text{tensor product. } 0.13
\end{align*}